練習問題解答例

＜第３章　流体の力学＞

３．１　式(3.3)を極座標形式で表せ。

解）微分変数が r と θ となるように微分公式を用いる。

$$\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}=\frac{\partial \rho u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial \rho u}{\partial \theta}\frac{\partial \theta}{\partial x}+\frac{\partial \rho v}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial \rho v}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$=\left(\rho\frac{\partial u}{\partial r}+u\frac{\partial \rho}{\partial r}\right)\frac{\partial r}{\partial x}+\left(\rho\frac{\partial u}{\partial \theta}+u\frac{\partial \rho}{\partial \theta}\right)\frac{\partial \theta}{\partial x}+\left(\rho\frac{\partial v}{\partial r}+v\frac{\partial \rho}{\partial r}\right)\frac{\partial r}{\partial y}+\left(\rho\frac{\partial v}{\partial \theta}+v\frac{\partial \rho}{\partial \theta}\right)\frac{\partial \theta}{\partial y}$$

この各項が極座標で表せればよい。

１次微分を整理しておく。

$r=\sqrt{x^2+y^2}$ より

$$\frac{\partial r}{\partial x}=\frac{2x}{2\sqrt{x^2+y^2}}=\frac{x}{\sqrt{x^2+y^2}}=\frac{x}{r}=\cos\theta$$

$$\frac{\partial r}{\partial y}=\frac{2y}{2\sqrt{x^2+y^2}}=\frac{y}{\sqrt{x^2+y^2}}=\frac{y}{r}=\sin\theta$$

$\theta=Arc\tan\dfrac{y}{x}$ より

$$\frac{\partial \theta}{\partial x}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2+y^2}=-\frac{y}{r^2}=-\frac{1}{r}\frac{y}{r}=-\frac{\sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)=\frac{x}{x^2+y^2}=\frac{x}{r^2}=\frac{1}{r}\frac{x}{r}=\frac{\cos\theta}{r}$$

$$u=\frac{\partial x}{\partial t}=\frac{\partial}{\partial t}r\cos\theta=r\frac{\partial}{\partial t}\cos\theta+\cos\theta\frac{\partial}{\partial t}r=-r\sin\theta\frac{\partial \theta}{\partial t}+\cos\theta\frac{\partial r}{\partial t}=-\sin\theta v_\theta+\cos\theta v_r$$

$$v=\frac{\partial y}{\partial t}=\frac{\partial}{\partial t}r\sin\theta=r\frac{\partial}{\partial t}\sin\theta+\sin\theta\frac{\partial}{\partial t}r=r\cos\theta\frac{\partial \theta}{\partial t}+\sin\theta\frac{\partial r}{\partial t}=\cos\theta v_\theta+\sin\theta v_r$$

ここで

$$v_r = \frac{\partial r}{\partial t}$$

$$v_\theta = r\frac{\partial \theta}{\partial t}$$

に注意。

$$\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}\left(-\sin\theta v_\theta + \cos\theta v_r\right) = -\sin\theta\frac{\partial v_\theta}{\partial r} + \cos\theta\frac{\partial v_r}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial}{\partial \theta}\left(-\sin\theta v_\theta + \cos\theta v_r\right) = -\cos\theta v_\theta - \sin\theta\frac{\partial v_\theta}{\partial \theta} - \sin\theta v_r + \cos\theta\frac{\partial v_r}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = \frac{\partial}{\partial r}\left(\cos\theta v_\theta + \sin\theta v_r\right) = \cos\theta\frac{\partial v_\theta}{\partial r} + \sin\theta\frac{\partial v_r}{\partial r}$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\cos\theta v_\theta + \sin\theta v_r\right) = -\sin\theta v_\theta + \cos\theta\frac{\partial v_\theta}{\partial \theta} + \cos\theta v_r + \sin\theta\frac{\partial v_r}{\partial \theta}$$

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = \frac{\partial \rho u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \rho u}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial \rho v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \rho v}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$= \left(\rho\frac{\partial u}{\partial r} + u\frac{\partial \rho}{\partial r}\right)\frac{\partial r}{\partial x} + \left(\rho\frac{\partial u}{\partial \theta} + u\frac{\partial \rho}{\partial \theta}\right)\frac{\partial \theta}{\partial x} + \left(\rho\frac{\partial v}{\partial r} + v\frac{\partial \rho}{\partial r}\right)\frac{\partial r}{\partial y} + \left(\rho\frac{\partial v}{\partial \theta} + v\frac{\partial \rho}{\partial \theta}\right)\frac{\partial \theta}{\partial y}$$

$$= \left\{\rho\left(-\sin\theta\frac{\partial v_\theta}{\partial r} + \cos\theta\frac{\partial v_r}{\partial r}\right) + \left(-\sin\theta v_\theta + \cos\theta v_r\right)\frac{\partial \rho}{\partial r}\right\}\cos\theta$$

$$+ \left\{\rho\left(-\cos\theta v_\theta - \sin\theta\frac{\partial v_\theta}{\partial \theta} - \sin\theta v_r + \cos\theta\frac{\partial v_r}{\partial \theta}\right) + \left(-\sin\theta v_\theta + \cos\theta v_r\right)\frac{\partial \rho}{\partial \theta}\right\}\left(-\frac{\sin\theta}{r}\right)$$

$$+ \left\{\rho\left(\cos\theta\frac{\partial v_\theta}{\partial r} + \sin\theta\frac{\partial v_r}{\partial r}\right) + \left(\cos\theta v_\theta + \sin\theta v_r\right)\frac{\partial \rho}{\partial r}\right\}\sin\theta$$

$$+ \left\{\rho\left(-\sin\theta v_\theta + \cos\theta\frac{\partial v_\theta}{\partial \theta} + \cos\theta v_r + \sin\theta\frac{\partial v_r}{\partial \theta}\right) + \left(\cos\theta v_\theta + \sin\theta v_r\right)\frac{\partial \rho}{\partial \theta}\right\}\left(\frac{\cos\theta}{r}\right)$$

$$= \left\{\left(-\rho\sin\theta\frac{\partial v_\theta}{\partial r} + \rho\cos\theta\frac{\partial v_r}{\partial r}\right) + \left(-\frac{\partial \rho}{\partial r}\sin\theta v_\theta + \frac{\partial \rho}{\partial r}\cos\theta v_r\right)\right\}\cos\theta$$

$$+ \left\{\left(-\rho\cos\theta v_\theta - \rho\sin\theta\frac{\partial v_\theta}{\partial \theta} - \rho\sin\theta v_r + \rho\cos\theta\frac{\partial v_r}{\partial \theta}\right) + \left(-\frac{\partial \rho}{\partial \theta}\sin\theta v_\theta + \frac{\partial \rho}{\partial \theta}\cos\theta v_r\right)\right\}\left(-\frac{\sin\theta}{r}\right)$$

$$+ \left\{\left(\rho\cos\theta\frac{\partial v_\theta}{\partial r} + \rho\sin\theta\frac{\partial v_r}{\partial r}\right) + \left(\frac{\partial \rho}{\partial r}\cos\theta v_\theta + \frac{\partial \rho}{\partial r}\sin\theta v_r\right)\right\}\sin\theta$$

$$+ \left\{\left(-\rho\sin\theta v_\theta + \rho\cos\theta\frac{\partial v_\theta}{\partial \theta} + \rho\cos\theta v_r + \rho\sin\theta\frac{\partial v_r}{\partial \theta}\right) + \left(\frac{\partial \rho}{\partial \theta}\cos\theta v_\theta + \frac{\partial \rho}{\partial \theta}\sin\theta v_r\right)\right\}\left(\frac{\cos\theta}{r}\right)$$

$$=-\rho\sin\theta\cos\theta\frac{\partial v_\theta}{\partial r}+\rho\cos^2\theta\frac{\partial v_r}{\partial r}-\frac{\partial\rho}{\partial r}\sin\theta\cos\theta v_\theta+\frac{\partial\rho}{\partial r}\cos^2\theta v_r$$

$$+\rho\frac{\sin\theta\cos\theta}{r}v_\theta+\rho\frac{\sin^2\theta}{r}\frac{\partial v_\theta}{\partial\theta}+\rho\frac{\sin^2\theta}{r}v_r-\rho\frac{\sin\theta\cos\theta}{r}\frac{\partial v_r}{\partial\theta}+\frac{\partial\rho}{\partial\theta}\frac{\sin^2\theta}{r}v_\theta-\frac{\partial\rho}{\partial\theta}\frac{\sin\theta\cos\theta}{r}v_r$$

$$+\rho\sin\theta\cos\theta\frac{\partial v_\theta}{\partial r}+\rho\sin^2\theta\frac{\partial v_r}{\partial r}+\frac{\partial\rho}{\partial r}\sin\theta\cos\theta v_\theta+\frac{\partial\rho}{\partial r}\sin^2\theta v_r$$

$$-\rho\frac{\sin\theta\cos\theta}{r}v_\theta+\rho\frac{\cos^2\theta}{r}\frac{\partial v_\theta}{\partial\theta}+\rho\frac{\cos^2\theta}{r}v_r+\rho\frac{\sin\theta\cos\theta}{r}\frac{\partial v_r}{\partial\theta}+\frac{\partial\rho}{\partial\theta}\frac{\cos^2\theta}{r}v_\theta+\frac{\partial\rho}{\partial\theta}\frac{\sin\theta\cos\theta}{r}v_r$$

$$=\frac{\partial\rho}{\partial r}\cos^2\theta v_r+\rho\frac{\sin^2\theta}{r}v_r-\frac{\partial\rho}{\partial\theta}\frac{\sin\theta\cos\theta}{r}v_r+\frac{\partial\rho}{\partial r}\sin^2\theta v_r+\rho\frac{\cos^2\theta}{r}v_r+\frac{\partial\rho}{\partial\theta}\frac{\sin\theta\cos\theta}{r}v_r$$

$$+\rho\cos^2\theta\frac{\partial v_r}{\partial r}+\rho\sin^2\theta\frac{\partial v_r}{\partial r}-\rho\frac{\sin\theta\cos\theta}{r}\frac{\partial v_r}{\partial\theta}+\rho\frac{\sin\theta\cos\theta}{r}\frac{\partial v_r}{\partial\theta}$$

$$-\frac{\partial\rho}{\partial r}\sin\theta\cos\theta v_\theta+\rho\frac{\sin\theta\cos\theta}{r}v_\theta+\frac{\partial\rho}{\partial\theta}\frac{\sin^2\theta}{r}v_\theta+\frac{\partial\rho}{\partial r}\sin\theta\cos\theta v_\theta-\rho\frac{\sin\theta\cos\theta}{r}v_\theta+\frac{\partial\rho}{\partial\theta}\frac{\cos^2\theta}{r}v_\theta$$

$$-\rho\sin\theta\cos\theta\frac{\partial v_\theta}{\partial r}+\rho\sin\theta\cos\theta\frac{\partial v_\theta}{\partial r}+\rho\frac{\sin^2\theta}{r}\frac{\partial v_\theta}{\partial\theta}+\rho\frac{\cos^2\theta}{r}\frac{\partial v_\theta}{\partial\theta}$$

$$=\frac{\partial\rho}{\partial r}v_r+\rho\frac{v_r}{r}+\rho\frac{\partial v_r}{\partial r}+\frac{\partial\rho}{\partial\theta}\frac{v_\theta}{r}+\rho\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}$$

$$=\frac{\partial\rho}{\partial r}v_r+\rho\frac{v_r}{r}+\rho\frac{\partial v_r}{\partial r}+\frac{\partial\rho}{\partial\theta}\frac{v_\theta}{r}+\rho\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}$$

$$=\frac{1}{r}\left(r\frac{\partial\rho}{\partial r}v_r+\rho v_r+r\rho\frac{\partial v_r}{\partial r}\right)+\frac{1}{r}\left(\frac{\partial\rho}{\partial\theta}v_\theta+\rho\frac{\partial v_\theta}{\partial\theta}\right)$$

$$=\frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho v_\theta)$$

これを式(3.3)に代入して

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho v_\theta)=0$$

３．２　連続の式(3.5)を導け。

解）２次元微小要素 $\Delta x\times\Delta y$ について考える。微小時間 Δt に左の面から流入する質量は

$\rho u \Delta y \Delta t$　　　　(3.2.A1)

x 方向に少し進んだ位置 $x+\Delta x$ である右の面から流出する質量は

$$\rho u \Delta y \Delta t + \frac{\partial}{\partial x}(\rho u \Delta y \Delta t)\Delta x \quad (3.2.A2)$$

同様に微小時間 Δt に下の面から流入する質量は

$\rho \upsilon \Delta x \Delta t$　　　　(3.2.A3)

y 方向に少し進んだ位置 $y+\Delta y$ である右の面から流出する質量は

$$\rho \upsilon \Delta x \Delta t + \frac{\partial}{\partial y}(\rho \upsilon \Delta x \Delta t)\Delta y \quad (3.2.A4)$$

これらの収支を取ったものは、

$$\rho u \Delta y \Delta t + \rho \upsilon \Delta x \Delta t - \left\{\rho u \Delta y \Delta t + \frac{\partial}{\partial x}(\rho u \Delta y \Delta t)\Delta x\right\} - \left\{\rho \upsilon \Delta x \Delta t + \frac{\partial}{\partial y}(\rho \upsilon \Delta x \Delta t)\Delta y\right\}$$

$$= -\left\{\frac{\partial}{\partial x}(\rho u \Delta y \Delta t)\Delta x\right\} - \left\{\frac{\partial}{\partial y}(\rho \upsilon \Delta x \Delta t)\Delta y\right\} = -\left\{\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho \upsilon)\right\}\Delta x \Delta y \Delta t \quad (3.2.A5)$$

これが微小時間 Δt における微小要素 $\Delta x \times \Delta y$ の密度の増加による質量の増加になるが、この質量増加は、

$$\frac{\partial \rho}{\partial t}\Delta t(\Delta x \Delta y) = \frac{\partial \rho}{\partial t}\Delta x \Delta y \Delta t \quad (3.2.A6)$$

よって、

$$\frac{\partial \rho}{\partial t}\Delta x \Delta y \Delta t = -\left\{\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho \upsilon)\right\}\Delta x \Delta y \Delta t \quad (3.2.A7)$$

これを整理すると、非定常でも定常でも、圧縮性でも非圧縮性でも、成立する連続の式が次のように得られる。

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho \upsilon)}{\partial y} = 0 \quad (3.3)$$

非定常であっても非圧縮性（密度一様）の流体については

$$\frac{\partial \rho}{\partial t} = 0 \quad (3.2.A8)$$

なので、

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0 \qquad (3.2.A9)$$

となり

$$\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0 \qquad (3.2.A10)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (3.5)$$

が得られる。

なお、式(3.3)を３次元流れの一般形とすると同様の議論によって

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0 \qquad (3.2.A11)$$

が得られるが、これは発散の定義により

$$\frac{\partial \rho}{\partial t}+\mathrm{div}(\rho\vec{v})=0 \qquad (3.2.A12)$$

と表される。

３.３　式(3.5)を円柱座標表示で導出せよ。

解）半径方向位置が r から $r+\Delta r$ 、中心角方向位置が θ から $\theta+\Delta\theta$ の２次元微小要素について考える。微小時間 Δt に半径方向位置 r の面から流入する質量は

$$\rho u_r(r\Delta\theta)\Delta t=\rho u_r r\Delta\theta\Delta t \qquad (3.3.A1)$$

ここで u_r [m/s]は流体要素の半径方向位置変化の時間微分。半径方向に少し進んだ位置 $r+\Delta r$ の面から流出する質量は

$$\rho u_r r\Delta\theta\Delta t+\frac{\partial}{\partial r}(\rho u_r r\Delta\theta\Delta t)\Delta r \qquad (3.3.A2)$$

同様に微小時間 Δt に中心角方向位置が θ の面から流入する質量は

$$\rho(ru_\theta)\Delta r\Delta t=\rho ru_\theta\Delta r\Delta t \qquad (3.3.A3)$$

ここで u_θ [rad/s]は流体要素の半径方向位置変化の時間微分。中心角方向に少し進んだ位置 $\theta+\Delta\theta$ の面から流出する質量は

$$\rho r u_\theta \Delta r \Delta t + \frac{\partial}{\partial \theta}(\rho r u_\theta \Delta r \Delta t)\Delta\theta \qquad (3.3.A4)$$

これらの収支を取ったものは、

$$\rho u_r r\Delta\theta\Delta t + \rho r u_\theta \Delta r\Delta t - \left\{\rho u_r r\Delta\theta\Delta t + \frac{\partial}{\partial r}(\rho u_r r\Delta\theta\Delta t)\Delta r\right\} - \left\{\rho r u_\theta\Delta r\Delta t + \frac{\partial}{\partial\theta}(\rho r u_\theta\Delta r\Delta t)\Delta\theta\right\}$$
$$= -\left\{\frac{\partial}{\partial r}(\rho u_r r\Delta\theta\Delta t)\Delta r\right\} - \left\{\frac{\partial}{\partial\theta}(\rho r u_\theta\Delta r\Delta t)\Delta\theta\right\} = -\left\{\frac{\partial}{\partial r}(\rho u_r r) + \frac{\partial}{\partial\theta}(\rho r u_\theta)\right\}\Delta r\Delta\theta\Delta t$$

(3.3.A5)

これが微小時間 Δt における微小要素 $\Delta r \times \Delta\theta$ （長方形ではないことに注意）の密度の増加による質量の増加になる。この部分の面積は、

$$\frac{1}{2}(r+\Delta r)^2\Delta\theta - \frac{1}{2}r^2\Delta\theta = \frac{1}{2}\left\{2r\Delta r + (\Delta r)^2\right\}\Delta\theta = r\Delta r\Delta\theta + \frac{1}{2}(\Delta r)^2\Delta\theta \qquad (3.3.A6)$$

だが、微小区間を考えるとき、高次の微小量は落とせるので、

$$r\Delta r\Delta\theta \qquad (3.3.A7)$$

となり、この部分の質量増加は、

$$\frac{\partial\rho}{\partial t}\Delta t(r\Delta r\Delta\theta) = \frac{\partial\rho}{\partial t} r\Delta r\Delta\theta\Delta t \qquad (3.3.A8)$$

よって、

$$\frac{\partial\rho}{\partial t} r\Delta r\Delta\theta\Delta t = -\left\{\frac{\partial}{\partial r}(\rho u_r r) + \frac{\partial}{\partial\theta}(\rho r u_\theta)\right\}\Delta r\Delta\theta\Delta t \qquad (3.3.A9)$$

これを整理すると、非定常でも定常でも、圧縮性でも非圧縮性でも、成立する連続の式が次のように得られる。

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\left\{\frac{\partial}{\partial r}(\rho u_r r) + \frac{\partial}{\partial\theta}(\rho r u_\theta)\right\} = 0 \qquad (3.3.A10)$$

非定常であっても非圧縮性（密度一様）の流体については

$$\frac{\partial\rho}{\partial t} = 0 \qquad (3.3.A11)$$

なので、

$$\frac{\partial}{\partial r}(\rho u_r r)+\frac{\partial}{\partial \theta}(\rho r u_\theta)=0 \qquad (3.3.A12)$$

となり

$$\rho\left\{\frac{\partial}{\partial r}(u_r r)+\frac{\partial}{\partial \theta}(r u_\theta)\right\}=0 \qquad (3.3.A13)$$

$$\frac{\partial}{\partial r}(u_r r)+\frac{\partial}{\partial \theta}(r u_\theta)=0 \qquad (3.3.A14)$$

$$r\frac{\partial u_r}{\partial r}+\frac{\partial r}{\partial r}u_r+\frac{\partial r}{\partial \theta}u_\theta+r\frac{\partial u_\theta}{\partial \theta}=0 \qquad (3.3.A15)$$

ここで、$\frac{\partial r}{\partial r}=1$ 、$\frac{\partial r}{\partial \theta}=0$ なので、

$$r\frac{\partial u_r}{\partial r}+u_r+r\frac{\partial u_\theta}{\partial \theta}=0 \qquad (3.3.A15)$$

が得られる。これが円筒座標系の連続の式である。これを直交座標に変換する。
なお、式(3.3.A15)は以下のようにも書ける。

$$\frac{\partial}{\partial r}(r u_r)+\frac{\partial}{\partial \theta}(r u_\theta)=0 \qquad (3.3.A16)$$

１次微分を整理しておく。

$x=r\cos\theta$ より

$$\frac{\partial x}{\partial r}=\cos\theta=\frac{x}{r}=\frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial x}{\partial \theta}=-r\sin\theta=-r\frac{y}{r}=-y$$

$y=r\sin\theta$ より

$$\frac{\partial y}{\partial r}=\sin\theta=\frac{y}{r}=\frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial y}{\partial \theta}=r\cos\theta=r\frac{x}{r}=x$$

$$u_r=\frac{\partial r}{\partial t}=\frac{\partial}{\partial t}\sqrt{x^2+y^2}=\frac{\partial}{\partial x}\sqrt{x^2+y^2}\frac{\partial x}{\partial t}+\frac{\partial}{\partial y}\sqrt{x^2+y^2}\frac{\partial y}{\partial t}$$

$$=\frac{1}{2\sqrt{x^2+y^2}}(2x)\frac{\partial x}{\partial t}+\frac{1}{2\sqrt{x^2+y^2}}(2y)\frac{\partial y}{\partial t}$$

$$=\frac{x}{\sqrt{x^2+y^2}}u+\frac{y}{\sqrt{x^2+y^2}}v$$

$$u_\theta=\frac{\partial \theta}{\partial t}=\frac{\partial}{\partial t}Arc\tan\frac{y}{x}=\frac{\partial}{\partial x}\left(Arc\tan\frac{y}{x}\right)\frac{\partial x}{\partial t}+\frac{\partial}{\partial y}\left(Arc\tan\frac{y}{x}\right)\frac{\partial y}{\partial t}$$

$$=\left(\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)\right)\frac{\partial x}{\partial t}+\left(\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)\right)\frac{\partial y}{\partial t}$$

$$=\left(\frac{-y}{x^2+y^2}\right)\frac{\partial x}{\partial t}+\left(\frac{x}{x^2+y^2}\right)\frac{\partial y}{\partial t}$$

$$=\left(\frac{-y}{x^2+y^2}\right)u+\left(\frac{x}{x^2+y^2}\right)v$$

$$\frac{\partial u_r}{\partial r}=\frac{\partial u_r}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u_r}{\partial y}\frac{\partial y}{\partial r}$$

$$=\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}u+\frac{y}{\sqrt{x^2+y^2}}v\right)\frac{\partial x}{\partial r}+\frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}u+\frac{y}{\sqrt{x^2+y^2}}v\right)\frac{\partial y}{\partial r}$$

$$=\left(\frac{1}{\sqrt{x^2+y^2}}u+\frac{x(2x)}{-2\sqrt{x^2+y^2}^3}u+\frac{x}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial x}+\frac{y(2x)}{-2\sqrt{x^2+y^2}^3}v+\frac{y}{\sqrt{x^2+y^2}}\frac{\partial v}{\partial x}\right)\frac{\partial x}{\partial r}$$

$$+\left(\frac{x(2y)}{-2\sqrt{x^2+y^2}^3}u+\frac{x}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial y}+\frac{1}{\sqrt{x^2+y^2}}v+\frac{y(2y)}{-2\sqrt{x^2+y^2}^3}v+\frac{y}{\sqrt{x^2+y^2}}\frac{\partial v}{\partial y}\right)\frac{\partial y}{\partial r}$$

$$=\left(\frac{u}{\sqrt{x^2+y^2}}-\frac{x^2}{\sqrt{x^2+y^2}^3}u+\frac{x}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial x}-\frac{xy}{\sqrt{x^2+y^2}^3}v+\frac{y}{\sqrt{x^2+y^2}}\frac{\partial v}{\partial x}\right)$$

$$\times\frac{x}{\sqrt{x^2+y^2}}$$

$$+\left(-\frac{xy}{\sqrt{x^2+y^2}^3}u+\frac{x}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial y}+\frac{v}{\sqrt{x^2+y^2}}-\frac{y^2}{\sqrt{x^2+y^2}^3}v+\frac{y}{\sqrt{x^2+y^2}}\frac{\partial v}{\partial y}\right)$$

$$\times\frac{y}{\sqrt{x^2+y^2}}$$

$$=\left(\frac{xu}{x^2+y^2}-\frac{x^3}{(x^2+y^2)^2}u+\frac{x^2}{x^2+y^2}\frac{\partial u}{\partial x}-\frac{x^2y}{(x^2+y^2)^2}v+\frac{xy}{x^2+y^2}\frac{\partial v}{\partial x}\right)$$

$$+\left(-\frac{xy^2}{(x^2+y^2)^2}u+\frac{xy}{x^2+y^2}\frac{\partial u}{\partial y}+\frac{yv}{x^2+y^2}-\frac{y^3}{(x^2+y^2)^2}v+\frac{y^2}{x^2+y^2}\frac{\partial v}{\partial y}\right)$$

$$=\left(\frac{x^3u+xy^2u}{(x^2+y^2)^2}-\frac{x^3u}{(x^2+y^2)^2}+\frac{x^2}{x^2+y^2}\frac{\partial u}{\partial x}-\frac{x^2y}{(x^2+y^2)^2}v+\frac{xy}{x^2+y^2}\frac{\partial v}{\partial x}\right)$$

$$+\left(-\frac{xy^2}{(x^2+y^2)^2}u+\frac{xy}{x^2+y^2}\frac{\partial u}{\partial y}+\frac{x^2yv+y^3v}{(x^2+y^2)^2}-\frac{y^3v}{(x^2+y^2)^2}+\frac{y^2}{x^2+y^2}\frac{\partial v}{\partial y}\right)$$

$$=\frac{x^2}{x^2+y^2}\frac{\partial u}{\partial x}+\frac{xy}{x^2+y^2}+\frac{\partial v}{\partial x}\frac{xy}{x^2+y^2}\frac{\partial u}{\partial y}+\frac{y^2}{x^2+y^2}\frac{\partial v}{\partial y}$$

$$\frac{\partial u_\theta}{\partial\theta}=\frac{\partial u_\theta}{\partial x}\frac{\partial x}{\partial\theta}+\frac{\partial u_\theta}{\partial y}\frac{\partial y}{\partial\theta}$$

$$=\frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}u+\frac{x}{x^2+y^2}v\right)\frac{\partial x}{\partial\theta}+\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}u+\frac{x}{x^2+y^2}v\right)\frac{\partial y}{\partial\theta}$$

$$=\left(\frac{-y(2x)}{-(x^2+y^2)^2}u+\frac{-y}{x^2+y^2}\frac{\partial u}{\partial x}+\frac{1}{x^2+y^2}v+\frac{x(2x)}{-(x^2+y^2)^2}v+\frac{x}{x^2+y^2}\frac{\partial v}{\partial x}\right)\frac{\partial x}{\partial\theta}$$

$$+\left(\frac{-1}{x^2+y^2}u+\frac{-y(2y)}{-(x^2+y^2)^2}u+\frac{-y}{x^2+y^2}\frac{\partial u}{\partial y}+\frac{x(2y)}{-(x^2+y^2)^2}v+\frac{x}{x^2+y^2}\frac{\partial v}{\partial y}\right)\frac{\partial y}{\partial\theta}$$

$$=\left(\frac{-y(2x)}{-(x^2+y^2)^2}u+\frac{-y}{x^2+y^2}\frac{\partial u}{\partial x}+\frac{1}{x^2+y^2}v+\frac{x(2x)}{-(x^2+y^2)^2}v+\frac{x}{x^2+y^2}\frac{\partial v}{\partial x}\right)(-y)$$

$$+\left(\frac{-1}{x^2+y^2}u+\frac{-y(2y)}{-\left(x^2+y^2\right)^2}u+\frac{-y}{x^2+y^2}\frac{\partial u}{\partial y}+\frac{x(2y)}{-\left(x^2+y^2\right)^2}v+\frac{x}{x^2+y^2}\frac{\partial v}{\partial y}\right)x$$

$$=\frac{-2xy^2u}{\left(x^2+y^2\right)^2}+\frac{y^2}{x^2+y^2}\frac{\partial u}{\partial x}+\frac{-yv}{x^2+y^2}+\frac{2x^2yv}{\left(x^2+y^2\right)^2}+\frac{-xy}{x^2+y^2}\frac{\partial v}{\partial x}$$

$$+\frac{-x}{x^2+y^2}u+\frac{-2xy^2}{-\left(x^2+y^2\right)^2}u+\frac{-xy}{x^2+y^2}\frac{\partial u}{\partial y}+\frac{2x^2y}{-\left(x^2+y^2\right)^2}v+\frac{x^2}{x^2+y^2}\frac{\partial v}{\partial y}$$

$$=\frac{-2xy^2u}{\left(x^2+y^2\right)^2}+\frac{y^2}{x^2+y^2}\frac{\partial u}{\partial x}+\frac{-x^2y-y^3}{\left(x^2+y^2\right)^2}v+\frac{2x^2yv}{\left(x^2+y^2\right)^2}+\frac{-xy}{x^2+y^2}\frac{\partial v}{\partial x}$$

$$+\frac{-x^3-xy^2}{\left(x^2+y^2\right)^2}u+\frac{2xy^2u}{\left(x^2+y^2\right)^2}+\frac{-xy}{x^2+y^2}\frac{\partial u}{\partial y}+\frac{-2x^2yv}{\left(x^2+y^2\right)^2}+\frac{x^2}{x^2+y^2}\frac{\partial v}{\partial y}$$

$$=\frac{y^2}{x^2+y^2}\frac{\partial u}{\partial x}+\frac{-x^2y-y^3}{\left(x^2+y^2\right)^2}v+\frac{-xy}{x^2+y^2}\frac{\partial v}{\partial x}$$

$$+\frac{-x^3-xy^2}{\left(x^2+y^2\right)^2}u+\frac{-xy}{x^2+y^2}\frac{\partial u}{\partial y}+\frac{x^2}{x^2+y^2}\frac{\partial v}{\partial y}$$

これらを式(3.3.A15)に代入。

$$\sqrt{x^2+y^2}\left(\frac{x^2}{x^2+y^2}\frac{\partial u}{\partial x}+\cancel{\frac{xy}{x^2+y^2}\frac{\partial v}{\partial x}}+\cancel{\frac{xy}{x^2+y^2}\frac{\partial u}{\partial y}}+\frac{y^2}{x^2+y^2}\frac{\partial v}{\partial y}\right)$$

$$+\frac{x}{\sqrt{x^2+y^2}}u+\frac{y}{\sqrt{x^2+y^2}}v$$

$$+\sqrt{x^2+y^2}\left(\frac{y^2}{x^2+y^2}\frac{\partial u}{\partial x}+\frac{-x^2y-y^3}{\left(x^2+y^2\right)^2}v+\cancel{\frac{-xy}{x^2+y^2}\frac{\partial v}{\partial x}}\right.$$

$$\left.+\frac{-x^3-xy^2}{\left(x^2+y^2\right)^2}u+\cancel{\frac{-xy}{x^2+y^2}\frac{\partial u}{\partial y}}+\frac{x^2}{x^2+y^2}\frac{\partial v}{\partial y}\right)=0$$

$$\sqrt{x^2+y^2}\left(\frac{x^2+y^2}{x^2+y^2}\frac{\partial u}{\partial x}+\frac{x^2+y^2}{x^2+y^2}\frac{\partial v}{\partial y}\right)$$

$$+\frac{x}{\sqrt{x^2+y^2}}u+\frac{y}{\sqrt{x^2+y^2}}v+\sqrt{x^2+y^2}\left(\frac{-y\left(x^2+y^2\right)}{\left(x^2+y^2\right)^2}v+\frac{-x\left(x^2+y^2\right)}{\left(x^2+y^2\right)^2}u\right)=0$$

$$\sqrt{x^2+y^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{x}{\sqrt{x^2+y^2}}u+\frac{y}{\sqrt{x^2+y^2}}v+\sqrt{x^2+y^2}\frac{\left(-yv+-xu\right)}{x^2+y^2}=0$$

$$\sqrt{x^2+y^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{x}{\sqrt{x^2+y^2}}u+\frac{y}{\sqrt{x^2+y^2}}v-\frac{y}{\sqrt{x^2+y^2}}v-\frac{x}{\sqrt{x^2+y^2}}u=0$$

$$\sqrt{x^2+y^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

式(3.5)が導かれた。

３．４　式(3.16)を導出せよ。

解）微小部分に作用する慣性力、圧力による力、応力による力、体積力のバランスを取る。

x 方向については、慣性力は式(3.10)、圧力は式(3.12)、応力は式(3.14)、体積力は $F_x(\rho\Delta x\Delta y)$ で表されるので、

$$\rho\Delta x\Delta y\left(\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}u+\frac{\partial u}{\partial y}\upsilon\right)=-\frac{\partial p}{\partial x}\Delta x\Delta y+\left(\frac{\partial}{\partial x}\sigma_{xx}\Delta x\right)\Delta y+\left(\frac{\partial}{\partial y}\tau_{yx}\Delta y\right)\Delta x+F_x\left(\rho\Delta x\Delta y\right)$$

両辺を $\Delta x\Delta y$ で割って

$$\rho\left(\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}u+\frac{\partial u}{\partial y}\upsilon\right)=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\sigma_{xx}+\frac{\partial}{\partial y}\tau_{yx}+\rho F_x$$

y 方向については、慣性力は式(3.11)、圧力は式(3.13)、応力は式(3.15)、体積力は $F_y(\rho\Delta x\Delta y)$ で表されるので、

$$\rho\Delta x\Delta y\left(\frac{\partial \upsilon}{\partial t}+\frac{\partial \upsilon}{\partial x}u+\frac{\partial \upsilon}{\partial y}\upsilon\right)=-\frac{\partial p}{\partial y}\Delta x\Delta y+\left(\frac{\partial}{\partial y}\sigma_{yy}\Delta y\right)\Delta x+\left(\frac{\partial}{\partial x}\tau_{xy}\Delta x\right)\Delta y+F_y(\rho\Delta x\Delta y)$$

両辺を $\Delta x\Delta y$ で割って

$$\rho\left(\frac{\partial \upsilon}{\partial t}+\frac{\partial \upsilon}{\partial x}u+\frac{\partial \upsilon}{\partial y}\upsilon\right)=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial y}\sigma_{yy}+\frac{\partial}{\partial x}\tau_{xy}+\rho F_y$$

式(3.16)が導かれた。

３.５　３次元の連続の式を求めよ。

解）微小区間 $\Delta x\Delta y\Delta z$ に、x 軸、y 軸、z 軸方向の流れによって微小時間 Δt 出入りする質量を考える。このとき、各方向の流速を u, v, w とする。

x 軸方向の流れによって $x=x$ の面から流入する流体の体積は、速度 u で面積 $\Delta y\Delta z$ を通って時間 Δt の間に流れ込む量なので、$u\Delta y\Delta z\Delta t$ となる。質量は、これに密度 ρ をかけて得られ、$\rho u\Delta y\Delta z\Delta t$ となる。

x 軸方向の流れによって $x=x+\Delta x$ の面から流出する流体の質量は、$x=x$ の面を通過する流量が、Δx 進む間に微少量だけ変化していることを考慮して、$\rho u\Delta y\Delta z\Delta t+\frac{\partial}{\partial x}(\rho u\Delta y\Delta z\Delta t)\Delta x$ となる。

よって、x 軸方向の流れによって正味で微小区間 $\Delta x\Delta y\Delta z$ に蓄積した量は

$$\rho u\Delta y\Delta z\Delta t-\left\{\rho u\Delta y\Delta z\Delta t+\frac{\partial}{\partial x}(\rho u\Delta y\Delta z\Delta t)\Delta x\right\}=-\frac{\partial}{\partial x}(\rho u\Delta y\Delta z\Delta t)\Delta x$$

である。同様にして、y 軸方向の流れによって正味で微小区間 $\Delta x\Delta y\Delta z$ に蓄積した量は

$$\rho v\Delta x\Delta z\Delta t-\left\{\rho v\Delta x\Delta z\Delta t+\frac{\partial}{\partial y}(\rho v\Delta x\Delta z\Delta t)\Delta y\right\}=-\frac{\partial}{\partial y}(\rho v\Delta x\Delta z\Delta t)\Delta y$$

、

z 軸方向の流れによって正味で微小区間 $\Delta x\Delta y\Delta z$ に蓄積した量は

$$\rho w\Delta x\Delta y\Delta t-\left\{\rho w\Delta x\Delta y\Delta t+\frac{\partial}{\partial z}(\rho w\Delta x\Delta y\Delta t)\Delta z\right\}=-\frac{\partial}{\partial z}(\rho w\Delta x\Delta y\Delta t)\Delta z$$

である。これらの蓄積量の和の分だけ、微小区間 $\Delta x\Delta y\Delta z$ 内の質量が微小時間 Δt の間に増加

する。微小区間 $\Delta x\Delta y\Delta z$ の質量は体積 $\Delta x\Delta y\Delta z$ に密度 ρ をかけて得られ、$\rho\Delta x\Delta y\Delta z$ となるが、

この微小時間 Δt の間の増加量は、

$$\frac{\partial}{\partial t}(\rho\Delta x\Delta y\Delta z)\Delta t$$

となるので、物質収支をとって、

$$\frac{\partial}{\partial t}(\rho\Delta x\Delta y\Delta z)\Delta t=-\frac{\partial}{\partial x}(\rho u\Delta y\Delta z\Delta t)\Delta x-\frac{\partial}{\partial y}(\rho v\Delta x\Delta z\Delta t)\Delta y-\frac{\partial}{\partial z}(\rho w\Delta x\Delta y\Delta t)\Delta z$$

$\Delta x,\Delta y,\Delta z,\Delta t$ は定数なので微分の前に出して

$$\Delta x\Delta y\Delta z\Delta t\frac{\partial}{\partial t}(\rho)=-\Delta x\Delta y\Delta z\Delta t\frac{\partial}{\partial x}(\rho u)-\Delta x\Delta y\Delta z\Delta t\frac{\partial}{\partial y}(\rho v)-\Delta x\Delta y\Delta z\Delta t\frac{\partial}{\partial z}(\rho w)$$

さらに両辺を $\Delta x\Delta y\Delta z\Delta t$ で割って

$$\frac{\partial}{\partial t}(\rho)=-\frac{\partial}{\partial x}(\rho u)-\frac{\partial}{\partial y}(\rho v)-\frac{\partial}{\partial z}(\rho w)$$

これを整理して、

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0$$

これが求める３次元の連続の式である。

３．６　オイラーの運動方程式を２次元・定常流について導け。

解）式(3.18)より

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+F_x$$

$$\frac{\partial \upsilon}{\partial t}+u\frac{\partial \upsilon}{\partial x}+\upsilon\frac{\partial \upsilon}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 \upsilon}{\partial x^2}+\frac{\partial^2 \upsilon}{\partial y^2}\right)+F_y \qquad (3.18)$$

定常状態では

$$\frac{\partial u}{\partial t}=\frac{\partial v}{\partial t}=0$$

また、粘性の影響が無視できるので

$$\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\nu\left(\frac{\partial^2 \upsilon}{\partial x^2}+\frac{\partial^2 \upsilon}{\partial y^2}\right)=0$$

これらを代入して

$$u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+F_x$$

$$u\frac{\partial \upsilon}{\partial x}+\upsilon\frac{\partial \upsilon}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+F_y \qquad (3.18)$$

これが、2 次元・定常流についてのオイラーの運動方程式である。

３．７　非圧縮性で粘性係数が一定の流体の場合の運動方程式(3.18)を導け。

解）x 方向について示す。

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\rho F_x \qquad (3.16)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{\rho}\frac{\partial \sigma_{xx}}{\partial x}+\frac{1}{\rho}\frac{\partial \tau_{yx}}{\partial y}+F_x$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{\rho}\frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x}-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}\right)\right]+\frac{1}{\rho}\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial \upsilon}{\partial x}\right)\right]+F_x$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{\rho}\left\{\frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x}-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial \upsilon}{\partial x}\right)\right]\right\}+F_x$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{\rho}\left\{\frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x}-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial \upsilon}{\partial x}\right)\right]\right\}+F_x$$

ここまでは任意の流体について成立。

粘性係数が一定の場合

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left\{\frac{\partial}{\partial x}\left[2\frac{\partial u}{\partial x}-\frac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}+\frac{\partial \upsilon}{\partial x}\right]\right\}+F_x$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left\{\left[2\frac{\partial^2 u}{\partial x^2}-\frac{2}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}\right)\right]+\left[\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 \upsilon}{\partial x\partial y}\right]\right\}+F_x$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left\{2\frac{\partial^2 u}{\partial x^2}-\frac{2}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}\right)+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 \upsilon}{\partial x\partial y}\right\}+F_x$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x^2}-\frac{2}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}\right)+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 \upsilon}{\partial x\partial y}\right\}+F_x$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left\{\frac{\partial^2 u}{\partial x^2}-\frac{2}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}\right)+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 \upsilon}{\partial x\partial y}\right\}+F_x$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left\{\frac{\partial^2 u}{\partial x^2}-\frac{2}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}\right)+\frac{\partial^2 u}{\partial y^2}+\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}\right)\right\}+F_x$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{1}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial \upsilon}{\partial y}\right)\right\}+F_x$$

さらに非圧縮性（密度一定）の場合、式(3.5)によって

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+F_x$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+F_x \tag{3.18a}$$

x 方向と同様にして y 方向の式は

$$\frac{\partial \upsilon}{\partial t}+u\frac{\partial \upsilon}{\partial x}+\upsilon\frac{\partial \upsilon}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 \upsilon}{\partial x^2}+\frac{\partial^2 \upsilon}{\partial y^2}\right)+F_y \tag{3.18b}$$

なお、動粘性係数

$$\nu = \frac{\mu}{\rho} \tag{3.19}$$

を導入している。

3．8　式(3.21)を３次元に拡張せよ。

解）x 軸方向について考える。慣性力については、z 軸方向の流れによっても x 軸方向の慣性力が生じるので、 $w\frac{\partial u}{\partial z}$ の項を加える。また、z 軸方向に存在する x 軸方向速度の分布によっても剪断応力が生じるので、 $\nu\frac{\partial^2 u}{\partial z^2}$ の項を加える。他の項には影響がないので、

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + F_x$$

y 軸方向、z 軸方向についても同様に、

$$\frac{\partial \upsilon}{\partial t} + u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y} + w\frac{\partial \upsilon}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 \upsilon}{\partial x^2} + \frac{\partial^2 \upsilon}{\partial y^2} + \frac{\partial^2 \upsilon}{\partial z^2}\right) + F_y$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + \upsilon\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + F_z$$

3．9　ベルヌイの定理(3.31)を導け。

解）式(3.30)を積分する。

$$\frac{1}{2}d\left(u^2 + v^2\right) = -d\phi - \frac{1}{\rho}dp \tag{3.30}$$

直交する x 方向速度 u と y 方向速度 v 合成速度 q がなので、三平方の定理から

$$q^2 = u^2 + v^2$$

これを代入して

$$\frac{1}{2}d\left(q^2\right) = -d\phi - \frac{1}{\rho}dp$$

$$\frac{1}{2}d\left(q^2\right) + d\phi + \frac{1}{\rho}dp = 0$$

両辺を不定積分する。

$$\int \frac{1}{2} d\left(q^2\right) + \int d\phi + \int \frac{1}{\rho} dp = C$$

C は積分定数。第 1 項の 1/2 は定数なので積分の外に出せるが、密度 ρ は一般的には圧力に依存するので、積分の外には出せない。よって

$$\frac{1}{2} \int d\left(q^2\right) + \int d\phi + \int \frac{1}{\rho} dp = C$$

$$\frac{1}{2} q^2 + \phi + \int \frac{1}{\rho} dp = C \tag{3.31}$$

３.１０　深さ 1 m の水槽の側面最下部に直径 1 cm の円形の穴を空けた。流量いくらで水が出てくるか。

解）水は非圧縮性なので、式(3.32)を用いる。上面の水が下部から出てくる。

水の密度 ρ は 1000 kg/m³、上部の水については速度 q_1 = 0 m/s、高さ h_1=1 m、大気圧は $p_1 =$ 101.3 kPa、出てきた水については、速度 q_2 [m/s]、高さ h_2 =0 m、大気圧は $p_2 =$ 101.3 kPa なので、

$$\frac{1}{2} \rho {q_1}^2 + \rho g h_1 + p_1 = \frac{1}{2} \rho {q_2}^2 + \rho g h_2 + p_2$$

$$q_2 = \sqrt{{q_1}^2 + 2g\left(h_1 - h_2\right) + \frac{2}{\rho}\left(p_1 - p_2\right)}$$

$$= \sqrt{(0)^2 + 2(9.81)(1-0) + \frac{2}{(1000)}\left(101.3\times10^3 - 101.3\times10^3\right)}$$

$$= \sqrt{19.62} = 4.43 \text{ m/s}$$

今、穴の径が 1 cm なので、流量は穴の面積に流速をかけて

$$\frac{\pi}{4}(0.01)^2(4.43) = 3.48\times10^{-4}\ \text{m}^3/\text{s} = 348\ \ \text{cm}^3/\text{s}$$

３.１１　エネルギー式(3.40)を導け。

解）微小部分に関する熱収支を取る。x 方向の流れによって蓄積される熱量（式(3.35)）、x 方向の熱伝導によって蓄積される熱量（式(3.36)）、 y 方向についての対応する熱量（式(3.37)）、体積あたりの発熱量（式(3.38)）と、温度変化の熱量（式(3.39)）が釣り合うので、

$$c\rho\frac{\partial T}{\partial t}\Delta x\Delta y\Delta t$$

$$=-c\rho u\frac{\partial T}{\partial x}\Delta x\Delta y\Delta t+\lambda\frac{\partial^2 T}{\partial x^2}\Delta x\Delta y\Delta t-c\rho v\frac{\partial T}{\partial y}\Delta x\Delta y\Delta t+\lambda\frac{\partial^2 T}{\partial y^2}\Delta x\Delta y\Delta t+Q_v\Delta x\Delta y\Delta t$$

両辺を $\Delta x\Delta y\Delta t$ で割って

$$c\rho\frac{\partial T}{\partial t}=-c\rho u\frac{\partial T}{\partial x}+\lambda\frac{\partial^2 T}{\partial x^2}-c\rho v\frac{\partial T}{\partial y}+\lambda\frac{\partial^2 T}{\partial y^2}+Q_v$$

$$c\rho\frac{\partial T}{\partial t}+c\rho u\frac{\partial T}{\partial x}+c\rho v\frac{\partial T}{\partial y}=\lambda\frac{\partial^2 T}{\partial x^2}+\lambda\frac{\partial^2 T}{\partial y^2}+Q_v$$

$$c\rho\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=\lambda\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+Q_v \tag{3.40}$$

３．１２　式(3.40)を極座標で表せ。

解）式(3.14)は以下のように表される。

$$c\rho\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+\upsilon\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=\lambda\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+Q_v$$

ここで、極座標の表記は以下の通り。（付録参照）

$$u\frac{\partial s}{\partial x}+v\frac{\partial s}{\partial y}+w\frac{\partial s}{\partial z}=v_r\frac{\partial s}{\partial r}+\frac{v_\phi}{r}\frac{\partial s}{\partial \phi}+\frac{v_\theta}{r\sin\phi}\frac{\partial s}{\partial \theta}$$

$$\frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial y^2}+\frac{\partial^2 s}{\partial z^2}=\frac{1}{r^2}\left(r^2\frac{\partial s}{\partial r}\right)+\frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial s}{\partial \phi}\right)+\frac{1}{r^2\sin^2\phi}\frac{\partial^2 s}{\partial \theta^2}$$

よって

$$c\rho\left(\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{v_\phi}{r}\frac{\partial T}{\partial \phi}+\frac{v_\theta}{r\sin\phi}\frac{\partial T}{\partial \theta}\right)$$

$$=\lambda\left\{\frac{1}{r^2}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial T}{\partial \phi}\right)+\frac{1}{r^2\sin^2\phi}\frac{\partial^2 T}{\partial \theta^2}\right\}+Q_v$$

（付録）球座標への変換公式

$$x = r\sin\phi\cos\theta$$
$$y = r\sin\phi\sin\theta$$
$$z = r\cos\phi$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\phi = Arc\tan\left(\sqrt{x^2 + y^2}\Big/ z\right)$$

$$\theta = Arc\tan(y/x)$$

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2 + y^2 + z^2} = \frac{1}{2}\frac{2x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r} = \sin\phi\cos\theta$$

$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}\sqrt{x^2 + y^2 + z^2} = \frac{1}{2}\frac{2y}{\sqrt{x^2 + y^2 + z^2}} = \frac{y}{\sqrt{x^2 + y^2 + z^2}} = \frac{y}{r} = \sin\phi\sin\theta$$

$$\frac{\partial r}{\partial z} = \frac{\partial}{\partial z}\sqrt{x^2 + y^2 + z^2} = \frac{1}{2}\frac{2z}{\sqrt{x^2 + y^2 + z^2}} = \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{z}{r} = \cos\phi$$

$$\frac{\partial\phi}{\partial x} = \frac{1}{1 + \left(\sqrt{x^2 + y^2}\Big/ z\right)^2}\frac{2x}{\left(2\sqrt{x^2 + y^2}\right)z} = \frac{1}{1 + \left(x^2 + y^2\right)/z^2}\frac{x}{\left(\sqrt{x^2 + y^2}\right)z}$$

$$= \frac{z^2}{z^2 + \left(x^2 + y^2\right)}\frac{x}{\left(\sqrt{x^2 + y^2}\right)z} = \frac{1}{x^2 + y^2 + z^2}\frac{z}{\left(\sqrt{x^2 + y^2}\right)}x$$

$$= \frac{1}{\sqrt{x^2 + y^2 + z^2}}\frac{z}{\left(\sqrt{x^2 + y^2}\right)}\frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{1}{r}\frac{1}{\tan\phi}\frac{x}{r} = \frac{1}{r}\frac{1}{\tan\phi}\sin\phi\cos\theta$$

$$= \frac{1}{r}\frac{\cos\phi}{\sin\phi}\sin\phi\cos\theta = \frac{\cos\theta\cos\phi}{r}$$

$$\frac{\partial\phi}{\partial y} = \frac{1}{1 + \left(\sqrt{x^2 + y^2}\Big/ z\right)^2}\frac{2y}{\left(2\sqrt{x^2 + y^2}\right)z} = \frac{1}{1 + \left(x^2 + y^2\right)/z^2}\frac{y}{\left(\sqrt{x^2 + y^2}\right)z}$$

$$= \frac{z^2}{z^2 + \left(x^2 + y^2\right)}\frac{y}{\left(\sqrt{x^2 + y^2}\right)z} = \frac{1}{x^2 + y^2 + z^2}\frac{z}{\left(\sqrt{x^2 + y^2}\right)}y$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}} \frac{z}{\left(\sqrt{x^2+y^2}\right)} \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{1}{r} \frac{1}{\tan\phi} \frac{y}{r} = \frac{1}{r} \frac{1}{\tan\phi} \sin\phi \sin\theta$$

$$= \frac{1}{r} \frac{\cos\phi}{\sin\phi} \sin\phi \sin\theta = \frac{\cos\phi \sin\theta}{r}$$

$$\frac{\partial\phi}{\partial z} = \frac{1}{1+\left(\sqrt{x^2+y^2}/z\right)^2} \frac{-\sqrt{x^2+y^2}}{z^2} = \frac{z^2}{z^2+\left(x^2+y^2\right)} \frac{-\sqrt{x^2+y^2}}{z^2}$$

$$= \frac{-z}{z^2+\left(x^2+y^2\right)} \frac{\sqrt{x^2+y^2}}{z} = \frac{-z}{r^2} \tan\phi = -\frac{r\cos\phi}{r^2} \tan\phi = -\frac{r\cos\phi}{r^2} \frac{\sin\phi}{\cos\phi} = -\frac{\sin\phi}{r}$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+(y/x)^2} \frac{-y}{x^2} = \frac{-y}{x^2+y^2} = \frac{-r\sin\phi\sin\theta}{(r\sin\phi\cos\theta)^2+(r\sin\phi\sin\theta)^2} = \frac{-r\sin\phi\sin\theta}{(r\sin\phi)^2} = \frac{-\sin\theta}{r\sin\phi}$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+(y/x)^2} \frac{1}{x} = \frac{x}{x^2+y^2} = \frac{r\sin\phi\cos\theta}{(r\sin\phi\cos\theta)^2+(r\sin\phi\sin\theta)^2} = \frac{r\sin\phi\cos\theta}{(r\sin\phi)^2} = \frac{\cos\theta}{r\sin\phi}$$

$$\frac{\partial\theta}{\partial z} = 0$$

$$\frac{\partial x}{\partial r} = \sin\phi\cos\theta$$

$$\frac{\partial x}{\partial\phi} = r\cos\phi\cos\theta$$

$$\frac{\partial x}{\partial\theta} = r\sin\phi(-\sin\theta) = -r\sin\phi\sin\theta$$

$$\frac{\partial y}{\partial r} = \sin\phi\sin\theta$$

$$\frac{\partial y}{\partial\phi} = r\cos\phi\sin\theta$$

$$\frac{\partial y}{\partial\theta} = r\sin\phi\cos\theta$$

$$\frac{\partial z}{\partial r} = \cos\phi$$

$$\frac{\partial z}{\partial\phi} = r(-\sin\phi) = -r\sin\phi$$

$$\frac{\partial z}{\partial\theta} = 0$$

流れ速度の変換公式

$$u=\frac{\partial x}{\partial t}=\frac{\partial x}{\partial r}\frac{\partial r}{\partial t}+\frac{\partial x}{\partial \phi}\frac{\partial \phi}{\partial t}+\frac{\partial x}{\partial \theta}\frac{\partial \theta}{\partial t}=\sin\phi\cos\theta\frac{\partial r}{\partial t}+r\cos\phi\cos\theta\frac{\partial \phi}{\partial t}-r\sin\phi\sin\theta\frac{\partial \theta}{\partial t}$$

$$v=\frac{\partial y}{\partial t}=\frac{\partial y}{\partial r}\frac{\partial r}{\partial t}+\frac{\partial y}{\partial \phi}\frac{\partial \phi}{\partial t}+\frac{\partial y}{\partial \theta}\frac{\partial \theta}{\partial t}=\sin\phi\sin\theta\frac{\partial r}{\partial t}+r\cos\phi\sin\theta\frac{\partial \phi}{\partial t}+r\sin\phi\cos\theta\frac{\partial \theta}{\partial t}$$

$$w=\frac{\partial z}{\partial t}=\frac{\partial z}{\partial r}\frac{\partial r}{\partial t}+\frac{\partial z}{\partial \phi}\frac{\partial \phi}{\partial t}+\frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t}=\cos\phi\frac{\partial r}{\partial t}-r\sin\phi\frac{\partial \phi}{\partial t}$$

１次微分の変換公式

$$\frac{\partial s}{\partial x}=\frac{\partial s}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial s}{\partial \phi}\frac{\partial \phi}{\partial x}+\frac{\partial s}{\partial \theta}\frac{\partial \theta}{\partial x}=\sin\phi\cos\theta\frac{\partial s}{\partial r}+\frac{\cos\theta\cos\phi}{r}\frac{\partial s}{\partial \phi}+\frac{-\sin\theta}{r\sin\phi}\frac{\partial s}{\partial \theta}$$

$$\frac{\partial s}{\partial y}=\frac{\partial s}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial s}{\partial \phi}\frac{\partial \phi}{\partial y}+\frac{\partial s}{\partial \theta}\frac{\partial \theta}{\partial y}=\sin\phi\sin\theta\frac{\partial s}{\partial r}+\frac{\cos\phi\sin\theta}{r}\frac{\partial s}{\partial \phi}+\frac{\cos\theta}{r\sin\phi}\frac{\partial s}{\partial \theta}$$

$$\frac{\partial s}{\partial z}=\frac{\partial s}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial s}{\partial \phi}\frac{\partial \phi}{\partial z}+\frac{\partial s}{\partial \theta}\frac{\partial \theta}{\partial z}=\cos\phi\frac{\partial s}{\partial r}+\frac{-\sin\phi}{r}\frac{\partial s}{\partial \phi}$$

同じ変数での２次微分の変換公式

$$\frac{\partial^2 s}{\partial x^2}=\frac{\partial}{\partial x}\left(\sin\phi\cos\theta\frac{\partial s}{\partial r}+\frac{\cos\theta\cos\phi}{r}\frac{\partial s}{\partial \phi}+\frac{-\sin\theta}{r\sin\phi}\frac{\partial s}{\partial \theta}\right)$$

$$=\frac{\partial}{\partial r}\left(\sin\phi\cos\theta\frac{\partial s}{\partial r}+\frac{\cos\theta\cos\phi}{r}\frac{\partial s}{\partial \phi}+\frac{-\sin\theta}{r\sin\phi}\frac{\partial s}{\partial \theta}\right)\frac{\partial r}{\partial x}$$

$$+\frac{\partial}{\partial \phi}\left(\sin\phi\cos\theta\frac{\partial s}{\partial r}+\frac{\cos\theta\cos\phi}{r}\frac{\partial s}{\partial \phi}+\frac{-\sin\theta}{r\sin\phi}\frac{\partial s}{\partial \theta}\right)\frac{\partial \phi}{\partial x}$$

$$+\frac{\partial}{\partial \theta}\left(\sin\phi\cos\theta\frac{\partial s}{\partial r}+\frac{\cos\theta\cos\phi}{r}\frac{\partial s}{\partial \phi}+\frac{-\sin\theta}{r\sin\phi}\frac{\partial s}{\partial \theta}\right)\frac{\partial \theta}{\partial x}$$

$$=\left(\sin\phi\cos\theta\frac{\partial^2 s}{\partial r^2}+\frac{-\cos\theta\cos\phi}{r^2}\frac{\partial s}{\partial \phi}+\frac{\cos\theta\cos\phi}{r}\frac{\partial^2 s}{\partial r\partial \phi}+\frac{\sin\theta}{r^2\sin\phi}\frac{\partial s}{\partial \theta}+\frac{-\sin\theta}{r\sin\phi}\frac{\partial^2 s}{\partial r\partial \theta}\right)\sin\phi\cos\theta$$

$$+\left(\cos\phi\cos\theta\frac{\partial s}{\partial r}+\sin\phi\cos\theta\frac{\partial^2 s}{\partial r\partial \phi}+\frac{-\cos\theta\sin\phi}{r}\frac{\partial s}{\partial \phi}+\frac{\cos\theta\cos\phi}{r}\frac{\partial^2 s}{\partial \phi^2}\right.$$

$$\left.+\frac{\cos\phi\sin\theta}{r\sin^2\phi}\frac{\partial s}{\partial \theta}+\frac{-\sin\theta}{r\sin\phi}\frac{\partial^2 s}{\partial \phi\partial \theta}\right)\frac{\cos\theta\cos\phi}{r}$$

$$+\left(-\sin\phi\sin\theta\frac{\partial s}{\partial r}+\sin\phi\cos\theta\frac{\partial^2 s}{\partial r\partial \theta}-\frac{\sin\theta\cos\phi}{r}\frac{\partial s}{\partial \phi}+\frac{\cos\theta\cos\phi}{r}\frac{\partial^2 s}{\partial \phi\partial \theta}\right.$$

$$+\frac{-\cos\theta}{r\sin\phi}\frac{\partial s}{\partial\theta}+\frac{-\sin\theta}{r\sin\phi}\frac{\partial^2 s}{\partial\theta^2}\Bigg)\frac{-\sin\theta}{r\sin\phi}$$

$$=\Bigg(\sin^2\phi\cos^2\theta\frac{\partial^2 s}{\partial r^2}+\frac{-\cos^2\theta\sin\phi\cos\phi}{r^2}\frac{\partial s}{\partial\phi}+\frac{\cos^2\theta\sin\phi\cos\phi}{r}\frac{\partial^2 s}{\partial r\partial\phi}$$

$$+\frac{\sin\theta\cos\theta}{r^2}\frac{\partial s}{\partial\theta}+\frac{-\sin\theta\cos\theta}{r}\frac{\partial^2 s}{\partial r\partial\theta}\Bigg)$$

$$+\Bigg(\frac{\cos^2\theta\cos^2\phi}{r}\frac{\partial s}{\partial r}+\frac{\sin\phi\cos^2\theta\cos\phi}{r}\frac{\partial^2 s}{\partial r\partial\phi}+\frac{-\cos^2\theta\sin\phi\cos\phi}{r^2}\frac{\partial s}{\partial\phi}+\frac{\cos^2\theta\cos^2\phi}{r^2}\frac{\partial^2 s}{\partial\phi^2}$$

$$+\frac{\cos^2\phi\sin\theta\cos\theta}{r^2\sin^2\phi}\frac{\partial s}{\partial\theta}+\frac{-\sin\theta\cos\theta\cos\phi}{r^2\sin\phi}\frac{\partial^2 s}{\partial\phi\partial\theta}\Bigg)$$

$$+\Bigg(\frac{\sin^2\theta}{r}\frac{\partial s}{\partial r}+\frac{-\sin\theta\cos\theta}{r}\frac{\partial^2 s}{\partial r\partial\theta}+\frac{\sin^2\theta\cos\phi}{r^2\sin\phi}\frac{\partial s}{\partial\phi}+\frac{-\sin\theta\cos\theta\cos\phi}{r^2\sin\phi}\frac{\partial^2 s}{\partial\phi\partial\theta}$$

$$+\frac{\sin\theta\cos\theta}{r^2\sin^2\phi}\frac{\partial s}{\partial\theta}+\frac{\sin^2\theta}{r^2\sin^2\phi}\frac{\partial^2 s}{\partial\theta^2}\Bigg)$$

$$=\sin^2\phi\cos^2\theta\frac{\partial^2 s}{\partial r^2}+\frac{\cos^2\theta\cos^2\phi}{r^2}\frac{\partial^2 s}{\partial\phi^2}+\frac{\sin^2\theta}{r^2\sin^2\phi}\frac{\partial^2 s}{\partial\theta^2}$$

$$+\frac{\cos^2\theta\sin\phi\cos\phi}{r}\frac{\partial^2 s}{\partial r\partial\phi}+\frac{\sin\phi\cos^2\theta\cos\phi}{r}\frac{\partial^2 s}{\partial r\partial\phi}+\frac{-\sin\theta\cos\theta}{r}\frac{\partial^2 s}{\partial r\partial\theta}+\frac{-\sin\theta\cos\theta}{r}\frac{\partial^2 s}{\partial r\partial\theta}$$

$$+\frac{-\sin\theta\cos\theta\cos\phi}{r^2\sin\phi}\frac{\partial^2 s}{\partial\phi\partial\theta}+\frac{-\sin\theta\cos\theta\cos\phi}{r^2\sin\phi}\frac{\partial^2 s}{\partial\phi\partial\theta}$$

$$\frac{\cos^2\theta\cos^2\phi}{r}\frac{\partial s}{\partial r}+\frac{\sin^2\theta}{r}\frac{\partial s}{\partial r}+\frac{-\cos^2\theta\sin\phi\cos\phi}{r^2}\frac{\partial s}{\partial\phi}+\frac{-\cos^2\theta\sin\phi\cos\phi}{r^2}\frac{\partial s}{\partial\phi}+\frac{\sin^2\theta\cos\phi}{r^2\sin\phi}\frac{\partial s}{\partial\phi}$$

$$+\frac{\sin\theta\cos\theta}{r^2}\frac{\partial s}{\partial\theta}+\frac{\cos^2\phi\sin\theta\cos\theta}{r^2\sin^2\phi}\frac{\partial s}{\partial\theta}+\frac{\sin\theta\cos\theta}{r^2\sin^2\phi}\frac{\partial s}{\partial\theta}$$

$$=\sin^2\phi\cos^2\theta\frac{\partial^2 s}{\partial r^2}+\frac{\cos^2\theta\cos^2\phi}{r^2}\frac{\partial^2 s}{\partial\phi^2}+\frac{\sin^2\theta}{r^2\sin^2\phi}\frac{\partial^2 s}{\partial\theta^2}$$

$$+\frac{2\cos^2\theta\sin\phi\cos\phi}{r}\frac{\partial^2 s}{\partial r\partial\phi}+\frac{-2\sin\theta\cos\theta}{r}\frac{\partial^2 s}{\partial r\partial\theta}+\frac{-2\sin\theta\cos\theta\cos\phi}{r^2\sin\phi}\frac{\partial^2 s}{\partial\phi\partial\theta}$$

$$+\frac{\cos^2\theta\cos^2\phi+\sin^2\theta}{r}\frac{\partial s}{\partial r}+\frac{-2\cos^2\theta\sin^2\phi\cos\phi+\sin^2\theta\cos\phi}{r^2\sin\phi}\frac{\partial s}{\partial\phi}$$

$$+\frac{\sin\theta\cos\theta\sin^2\phi+\cos^2\phi\sin\theta\cos\theta+\sin\theta\cos\theta}{r^2\sin^2\phi}\frac{\partial s}{\partial\theta}$$

$$=\sin^2\phi\cos^2\theta\frac{\partial^2 s}{\partial r^2}+\frac{\cos^2\phi\cos^2\theta}{r^2}\frac{\partial^2 s}{\partial\phi^2}+\frac{\sin^2\theta}{r^2\sin^2\phi}\frac{\partial^2 s}{\partial\theta^2}$$

$$+\frac{2\sin\phi\cos\phi\cos^2\theta}{r}\frac{\partial^2 s}{\partial r\partial\phi}+\frac{-2\sin\theta\cos\theta}{r}\frac{\partial^2 s}{\partial r\partial\theta}+\frac{-2\cos\phi\sin\theta\cos\theta}{r^2\sin\phi}\frac{\partial^2 s}{\partial\phi\partial\theta}$$

$$+\frac{\cos^2\theta\cos^2\phi+\sin^2\theta}{r}\frac{\partial s}{\partial r}+\frac{-2\cos^2\theta\sin^2\phi\cos\phi+\sin^2\theta\cos\phi}{r^2\sin\phi}\frac{\partial s}{\partial \phi}+\frac{2\sin\theta\cos\theta}{r^2\sin^2\phi}\frac{\partial s}{\partial \theta}$$

$$\frac{\partial^2 s}{\partial y^2}=\frac{\partial}{\partial y}\left(\sin\phi\sin\theta\frac{\partial s}{\partial r}+\frac{\cos\phi\sin\theta}{r}\frac{\partial s}{\partial \phi}+\frac{\cos\theta}{r\sin\phi}\frac{\partial s}{\partial \theta}\right)$$

$$=\frac{\partial}{\partial r}\left(\sin\phi\sin\theta\frac{\partial s}{\partial r}+\frac{\cos\phi\sin\theta}{r}\frac{\partial s}{\partial \phi}+\frac{\cos\theta}{r\sin\phi}\frac{\partial s}{\partial \theta}\right)\frac{\partial r}{\partial y}$$

$$+\frac{\partial}{\partial \phi}\left(\sin\phi\sin\theta\frac{\partial s}{\partial r}+\frac{\cos\phi\sin\theta}{r}\frac{\partial s}{\partial \phi}+\frac{\cos\theta}{r\sin\phi}\frac{\partial s}{\partial \theta}\right)\frac{\partial \phi}{\partial y}$$

$$+\frac{\partial}{\partial \theta}\left(\sin\phi\sin\theta\frac{\partial s}{\partial r}+\frac{\cos\phi\sin\theta}{r}\frac{\partial s}{\partial \phi}+\frac{\cos\theta}{r\sin\phi}\frac{\partial s}{\partial \theta}\right)\frac{\partial \theta}{\partial y}$$

$$=\left(\sin\phi\sin\theta\frac{\partial^2 s}{\partial r^2}+\frac{-\cos\phi\sin\theta}{r^2}\frac{\partial s}{\partial \phi}+\frac{\cos\phi\sin\theta}{r}\frac{\partial^2 s}{\partial r\partial \phi}+\frac{-\cos\theta}{r^2\sin\phi}\frac{\partial s}{\partial \theta}+\frac{\cos\theta}{r\sin\phi}\frac{\partial^2 s}{\partial r\partial \theta}\right)\sin\phi\sin\theta$$

$$+\left(\cos\phi\sin\theta\frac{\partial s}{\partial r}+\sin\phi\sin\theta\frac{\partial^2 s}{\partial r\partial \phi}+\frac{-\sin\phi\sin\theta}{r}\frac{\partial s}{\partial \phi}+\frac{\cos\phi\sin\theta}{r}\frac{\partial^2 s}{\partial \phi^2}\right.$$

$$\left.+\frac{-\cos\theta\cos\phi}{r\sin^2\phi}\frac{\partial s}{\partial \theta}+\frac{\cos\theta}{r\sin\phi}\frac{\partial^2 s}{\partial \phi\partial \theta}\right)\frac{\cos\phi\sin\theta}{r}$$

$$+\left(\sin\phi\cos\theta\frac{\partial s}{\partial r}+\sin\phi\sin\theta\frac{\partial^2 s}{\partial r\partial \theta}+\frac{\cos\phi\cos\theta}{r}\frac{\partial s}{\partial \phi}+\frac{\cos\phi\sin\theta}{r}\frac{\partial^2 s}{\partial \phi\partial \theta}\right.$$

$$\left.+\frac{-\sin\theta}{r\sin\phi}\frac{\partial s}{\partial \theta}+\frac{\cos\theta}{r\sin\phi}\frac{\partial^2 s}{\partial \theta^2}\right)\frac{\cos\theta}{r\sin\phi}$$

$$=\left(\sin^2\phi\sin^2\theta\frac{\partial^2 s}{\partial r^2}+\frac{-\sin\phi\cos\phi\sin^2\theta}{r^2}\frac{\partial s}{\partial \phi}+\frac{\sin\phi\cos\phi\sin^2\theta}{r}\frac{\partial^2 s}{\partial r\partial \phi}\right.$$

$$\left.+\frac{-\sin\theta\cos\theta}{r^2}\frac{\partial s}{\partial \theta}+\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 s}{\partial r\partial \theta}\right)$$

$$+\left(\frac{\cos^2\phi\sin^2\theta}{r}\frac{\partial s}{\partial r}+\frac{\sin\phi\cos\phi\sin^2\theta}{r}\frac{\partial^2 s}{\partial r\partial \phi}+\frac{-\sin\phi\cos\phi\sin^2\theta}{r^2}\frac{\partial s}{\partial \phi}+\frac{\cos^2\phi\sin^2\theta}{r^2}\frac{\partial^2 s}{\partial \phi^2}\right.$$

$$\left.+\frac{-\cos^2\phi\sin\theta\cos\theta}{r^2\sin^2\phi}\frac{\partial s}{\partial \theta}+\frac{\cos\phi\sin\theta\cos\theta}{r^2\sin\phi}\frac{\partial^2 s}{\partial \phi\partial \theta}\right)$$

$$+\left(\frac{\cos^2\theta}{r}\frac{\partial s}{\partial r}+\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 s}{\partial r\partial \theta}+\frac{\cos\phi\cos^2\theta}{r^2\sin\phi}\frac{\partial s}{\partial \phi}+\frac{\cos\phi\sin\theta\cos\theta}{r^2\sin\phi}\frac{\partial^2 s}{\partial \phi\partial \theta}\right.$$

$$\left.+\frac{-\sin\theta\cos\theta}{r^2\sin^2\phi}\frac{\partial s}{\partial \theta}+\frac{\cos^2\theta}{r^2\sin^2\phi}\frac{\partial^2 s}{\partial \theta^2}\right)$$

$$=\sin^2\phi\sin^2\theta\frac{\partial^2 s}{\partial r^2}+\frac{\cos^2\phi\sin^2\theta}{r^2}\frac{\partial^2 s}{\partial \phi^2}+\frac{\cos^2\theta}{r^2\sin^2\phi}\frac{\partial^2 s}{\partial \theta^2}$$

$$+\frac{\sin\phi\cos\phi\sin^2\theta}{r}\frac{\partial^2 s}{\partial r\partial\phi}+\frac{\sin\phi\cos\phi\sin^2\theta}{r}\frac{\partial^2 s}{\partial r\partial\phi}+\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 s}{\partial r\partial\theta}+\frac{\sin\theta\cos\theta}{r}\frac{\partial^2 s}{\partial r\partial\theta}$$

$$+\frac{\cos\phi\sin\theta\cos\theta}{r^2\sin\phi}\frac{\partial^2 s}{\partial\phi\partial\theta}+\frac{\cos\phi\sin\theta\cos\theta}{r^2\sin\phi}\frac{\partial^2 s}{\partial\phi\partial\theta}$$

$$+\frac{\cos^2\phi\sin^2\theta}{r}\frac{\partial s}{\partial r}+\frac{\cos^2\theta}{r}\frac{\partial s}{\partial r}$$

$$+\frac{-\sin\phi\cos\phi\sin^2\theta}{r^2}\frac{\partial s}{\partial\phi}+\frac{-\sin\phi\cos\phi\sin^2\theta}{r^2}\frac{\partial s}{\partial\phi}+\frac{\cos\phi\cos^2\theta}{r^2\sin\phi}\frac{\partial s}{\partial\phi}$$

$$+\frac{-\sin\theta\cos\theta}{r^2}\frac{\partial s}{\partial\theta}+\frac{-\cos^2\phi\sin\theta\cos\theta}{r^2\sin^2\phi}\frac{\partial s}{\partial\theta}+\frac{-\sin\theta\cos\theta}{r^2\sin^2\phi}\frac{\partial s}{\partial\theta}$$

$$=\sin^2\phi\sin^2\theta\frac{\partial^2 s}{\partial r^2}+\frac{\cos^2\phi\sin^2\theta}{r^2}\frac{\partial^2 s}{\partial\phi^2}+\frac{\cos^2\theta}{r^2\sin^2\phi}\frac{\partial^2 s}{\partial\theta^2}$$

$$+\frac{2\sin\phi\cos\phi\sin^2\theta}{r}\frac{\partial^2 s}{\partial r\partial\phi}+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 s}{\partial r\partial\theta}+\frac{2\cos\phi\sin\theta\cos\theta}{r^2\sin\phi}\frac{\partial^2 s}{\partial\phi\partial\theta}$$

$$+\frac{\cos^2\phi\sin^2\theta+\cos^2\theta}{r}\frac{\partial s}{\partial r}+\frac{-2\sin^2\phi\cos\phi\sin^2\theta+\cos\phi\cos^2\theta}{r^2\sin\phi}\frac{\partial s}{\partial\phi}$$

$$+\frac{-2\sin\theta\cos\theta}{r^2\sin^2\phi}\frac{\partial s}{\partial\theta}$$

$$\frac{\partial^2 s}{\partial z^2}=\frac{\partial}{\partial z}\left(\cos\phi\frac{\partial s}{\partial r}+\frac{-\sin\phi}{r}\frac{\partial s}{\partial\phi}\right)$$

$$=\frac{\partial}{\partial r}\left(\cos\phi\frac{\partial s}{\partial r}+\frac{-\sin\phi}{r}\frac{\partial s}{\partial\phi}\right)\frac{\partial r}{\partial z}+\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial s}{\partial r}+\frac{-\sin\phi}{r}\frac{\partial s}{\partial\phi}\right)\frac{\partial\phi}{\partial z}+\frac{\partial}{\partial\theta}\left(\cos\phi\frac{\partial s}{\partial r}+\frac{-\sin\phi}{r}\frac{\partial s}{\partial\phi}\right)\frac{\partial\theta}{\partial z}$$

$$=\left(\cos\phi\frac{\partial^2 s}{\partial r^2}+\frac{\sin\phi}{r^2}\frac{\partial s}{\partial\phi}+\frac{-\sin\phi}{r}\frac{\partial^2 s}{\partial r\partial\phi}\right)\cos\phi$$

$$+\left(-\sin\phi\frac{\partial s}{\partial r}+\cos\phi\frac{\partial^2 s}{\partial r\partial\phi}+\frac{-\cos\phi}{r}\frac{\partial s}{\partial\phi}+\frac{-\sin\phi}{r}\frac{\partial^2 s}{\partial\phi^2}\right)\left(-\frac{\sin\phi}{r}\right)$$

$$+\frac{\partial}{\partial\theta}\left(\cos\phi\frac{\partial s}{\partial r}+\frac{-\sin\phi}{r}\frac{\partial s}{\partial\phi}\right)(0)$$

$$=\left(\cos^2\phi\frac{\partial^2 s}{\partial r^2}+\frac{\sin\phi\cos\phi}{r^2}\frac{\partial s}{\partial\phi}+\frac{-\sin\phi\cos\phi}{r}\frac{\partial^2 s}{\partial r\partial\phi}\right)$$

$$+\left(\frac{\sin^2\phi}{r}\frac{\partial s}{\partial r}+\frac{-\sin\phi\cos\phi}{r}\frac{\partial^2 s}{\partial r\partial\phi}+\frac{\sin\phi\cos\phi}{r^2}\frac{\partial s}{\partial\phi}+\frac{\sin^2\phi}{r^2}\frac{\partial^2 s}{\partial\phi^2}\right)$$

$$=\cos^2\phi\frac{\partial^2 s}{\partial r^2}+\frac{\sin^2\phi}{r^2}\frac{\partial^2 s}{\partial\phi^2}+\frac{-\sin\phi\cos\phi}{r}\frac{\partial^2 s}{\partial r\partial\phi}+\frac{-\sin\phi\cos\phi}{r}\frac{\partial^2 s}{\partial r\partial\phi}$$

$$+\frac{\sin^2\phi}{r}\frac{\partial s}{\partial r}+\frac{\sin\phi\cos\phi}{r^2}\frac{\partial s}{\partial\phi}+\frac{\sin\phi\cos\phi}{r^2}\frac{\partial s}{\partial\phi}$$

$$=\cos^2\phi\frac{\partial^2 s}{\partial r^2}+\frac{\sin^2\phi}{r^2}\frac{\partial^2 s}{\partial\phi^2}+\frac{-2\sin\phi\cos\phi}{r}\frac{\partial^2 s}{\partial r\partial\phi}+\frac{\sin^2\phi}{r}\frac{\partial s}{\partial r}+\frac{2\sin\phi\cos\phi}{r^2}\frac{\partial s}{\partial\phi}$$

移流項の計算

$$u\frac{\partial s}{\partial x}=\left(\sin\phi\cos\theta\frac{\partial r}{\partial t}+r\cos\phi\cos\theta\frac{\partial\phi}{\partial t}-r\sin\phi\sin\theta\frac{\partial\theta}{\partial t}\right)$$

$$\times\left(\sin\phi\cos\theta\frac{\partial s}{\partial r}+\frac{\cos\theta\cos\phi}{r}\frac{\partial s}{\partial\phi}+\frac{-\sin\theta}{r\sin\phi}\frac{\partial s}{\partial\theta}\right)$$

$$=\left(\sin\phi\cos\theta\frac{\partial r}{\partial t}+r\cos\phi\cos\theta\frac{\partial\phi}{\partial t}-r\sin\phi\sin\theta\frac{\partial\theta}{\partial t}\right)\sin\phi\cos\theta\frac{\partial s}{\partial r}$$

$$+\left(\sin\phi\cos\theta\frac{\partial r}{\partial t}+r\cos\phi\cos\theta\frac{\partial\phi}{\partial t}-r\sin\phi\sin\theta\frac{\partial\theta}{\partial t}\right)\frac{\cos\theta\cos\phi}{r}\frac{\partial s}{\partial\phi}$$

$$+\left(\sin\phi\cos\theta\frac{\partial r}{\partial t}+r\cos\phi\cos\theta\frac{\partial\phi}{\partial t}-r\sin\phi\sin\theta\frac{\partial\theta}{\partial t}\right)\left(\frac{-\sin\theta}{r\sin\phi}\right)\frac{\partial s}{\partial\theta}$$

$$=\left(\sin^2\phi\cos^2\theta\frac{\partial r}{\partial t}+r\sin\phi\cos\phi\cos^2\theta\frac{\partial\phi}{\partial t}-r\sin^2\phi\sin\theta\cos\theta\frac{\partial\theta}{\partial t}\right)\frac{\partial s}{\partial r}$$

$$+\left(\frac{\sin\phi\cos\phi\cos^2\theta}{r}\frac{\partial r}{\partial t}+\cos^2\phi\cos^2\theta\frac{\partial\phi}{\partial t}-\sin\phi\cos\phi\sin\theta\cos\theta\frac{\partial\theta}{\partial t}\right)\frac{\partial s}{\partial\phi}$$

$$+\left(\frac{-\sin\theta\cos\theta}{r}\frac{\partial r}{\partial t}+\frac{-\cos\phi\sin\theta\cos\theta}{\sin\phi}\frac{\partial\phi}{\partial t}+\sin^2\theta\frac{\partial\theta}{\partial t}\right)\frac{\partial s}{\partial\theta}$$

$$v\frac{\partial s}{\partial y}=\left(\sin\phi\sin\theta\frac{\partial r}{\partial t}+r\cos\phi\sin\theta\frac{\partial\phi}{\partial t}+r\sin\phi\cos\theta\frac{\partial\theta}{\partial t}\right)$$

$$\times\left(\sin\phi\sin\theta\frac{\partial s}{\partial r}+\frac{\cos\phi\sin\theta}{r}\frac{\partial s}{\partial\phi}+\frac{\cos\theta}{r\sin\phi}\frac{\partial s}{\partial\theta}\right)$$

$$=\left(\sin\phi\sin\theta\frac{\partial r}{\partial t}+r\cos\phi\sin\theta\frac{\partial\phi}{\partial t}+r\sin\phi\cos\theta\frac{\partial\theta}{\partial t}\right)\sin\phi\sin\theta\frac{\partial s}{\partial r}$$

$$+\left(\sin\phi\sin\theta\frac{\partial r}{\partial t}+r\cos\phi\sin\theta\frac{\partial\phi}{\partial t}+r\sin\phi\cos\theta\frac{\partial\theta}{\partial t}\right)\frac{\cos\phi\sin\theta}{r}\frac{\partial s}{\partial\phi}$$

$$+\left(\sin\phi\sin\theta\frac{\partial r}{\partial t}+r\cos\phi\sin\theta\frac{\partial\phi}{\partial t}+r\sin\phi\cos\theta\frac{\partial\theta}{\partial t}\right)\frac{\cos\theta}{r\sin\phi}\frac{\partial s}{\partial\theta}$$

$$=\left(\sin^2\phi\sin^2\theta\frac{\partial r}{\partial t}+r\sin\phi\cos\phi\sin^2\theta\frac{\partial\phi}{\partial t}+r\sin^2\phi\sin\theta\cos\theta\frac{\partial\theta}{\partial t}\right)\frac{\partial s}{\partial r}$$

$$+\left(\frac{\sin\phi\cos\phi\sin^2\theta}{r}\frac{\partial r}{\partial t}+\cos^2\phi\sin^2\theta\frac{\partial\phi}{\partial t}+\sin\phi\cos\phi\sin\theta\cos\theta\frac{\partial\theta}{\partial t}\right)\frac{\partial s}{\partial\phi}$$

$$+\left(\frac{\sin\theta\cos\theta}{r}\frac{\partial r}{\partial t}+\frac{\cos\phi\sin\theta\cos\theta}{\sin\phi}\frac{\partial\phi}{\partial t}+\cos^2\theta\frac{\partial\theta}{\partial t}\right)\frac{\partial s}{\partial\theta}$$

$$w\frac{\partial s}{\partial z}=\left(\cos\phi\frac{\partial r}{\partial t}-r\sin\phi\frac{\partial\phi}{\partial t}\right)\left(\cos\phi\frac{\partial s}{\partial r}+\frac{-\sin\phi}{r}\frac{\partial s}{\partial\phi}\right)$$

$$=\left(\cos\phi\frac{\partial r}{\partial t}-r\sin\phi\frac{\partial\phi}{\partial t}\right)\cos\phi\frac{\partial s}{\partial r}$$

$$+\left(\cos\phi\frac{\partial r}{\partial t}-r\sin\phi\frac{\partial\phi}{\partial t}\right)\frac{-\sin\phi}{r}\frac{\partial s}{\partial\phi}$$

$$=\left(\cos^2\phi\frac{\partial r}{\partial t}-r\sin\phi\cos\phi\frac{\partial\phi}{\partial t}\right)\frac{\partial s}{\partial r}$$

$$+\left(\frac{-\sin\phi\cos\phi}{r}\frac{\partial r}{\partial t}+\sin^2\phi\frac{\partial\phi}{\partial t}\right)\frac{\partial s}{\partial\phi}$$

$$u\frac{\partial s}{\partial x}+v\frac{\partial s}{\partial y}+w\frac{\partial s}{\partial z}$$

$$=\left(\sin^2\phi\cos^2\theta\frac{\partial r}{\partial t}+r\sin\phi\cos\phi\cos^2\theta\frac{\partial\phi}{\partial t}-r\sin^2\phi\sin\theta\cos\theta\frac{\partial\theta}{\partial t}\right)\frac{\partial s}{\partial r}$$

$$+\left(\frac{\sin\phi\cos\phi\cos^2\theta}{r}\frac{\partial r}{\partial t}+\cos^2\phi\cos^2\theta\frac{\partial\phi}{\partial t}-\sin\phi\cos\phi\sin\theta\cos\theta\frac{\partial\theta}{\partial t}\right)\frac{\partial s}{\partial\phi}$$

$$+\left(\frac{-\sin\theta\cos\theta}{r}\frac{\partial r}{\partial t}+\frac{-\cos\phi\sin\theta\cos\theta}{\sin\phi}\frac{\partial\phi}{\partial t}+\sin^2\theta\frac{\partial\theta}{\partial t}\right)\frac{\partial s}{\partial\theta}$$

$$+\left(\sin^2\phi\sin^2\theta\frac{\partial r}{\partial t}+r\sin\phi\cos\phi\sin^2\theta\frac{\partial\phi}{\partial t}+r\sin^2\phi\sin\theta\cos\theta\frac{\partial\theta}{\partial t}\right)\frac{\partial s}{\partial r}$$

$$+\left(\frac{\sin\phi\cos\phi\sin^2\theta}{r}\frac{\partial r}{\partial t}+\cos^2\phi\sin^2\theta\frac{\partial\phi}{\partial t}+\sin\phi\cos\phi\sin\theta\cos\theta\frac{\partial\theta}{\partial t}\right)\frac{\partial s}{\partial\phi}$$

$$+\left(\frac{\sin\theta\cos\theta}{r}\frac{\partial r}{\partial t}+\frac{\cos\phi\sin\theta\cos\theta}{\sin\phi}\frac{\partial\phi}{\partial t}+\cos^2\theta\frac{\partial\theta}{\partial t}\right)\frac{\partial s}{\partial\theta}$$

$$+\left(\cos^2\phi\frac{\partial r}{\partial t}-r\sin\phi\cos\phi\frac{\partial\phi}{\partial t}\right)\frac{\partial s}{\partial r}$$

$$+\left(\frac{-\sin\phi\cos\phi}{r}\frac{\partial r}{\partial t}+\sin^2\phi\frac{\partial\phi}{\partial t}\right)\frac{\partial s}{\partial\phi}$$

$$=\left(\sin^2\phi\cos^2\theta\frac{\partial r}{\partial t}+r\sin\phi\cos\phi\cos^2\theta\frac{\partial\phi}{\partial t}-r\sin^2\phi\sin\theta\cos\theta\frac{\partial\theta}{\partial t}\right)\frac{\partial s}{\partial r}$$

$$+\left(\sin^2\phi\sin^2\theta\frac{\partial r}{\partial t}+r\sin\phi\cos\phi\sin^2\theta\frac{\partial\phi}{\partial t}+r\sin^2\phi\sin\theta\cos\theta\frac{\partial\theta}{\partial t}\right)\frac{\partial s}{\partial r}$$

$$+\left(\cos^2\phi\frac{\partial r}{\partial t}-r\sin\phi\cos\phi\frac{\partial\phi}{\partial t}\right)\frac{\partial s}{\partial r}$$

$$+\left(\frac{\sin\phi\cos\phi\cos^2\theta}{r}\frac{\partial r}{\partial t}+\cos^2\phi\cos^2\theta\frac{\partial\phi}{\partial t}-\sin\phi\cos\phi\sin\theta\cos\theta\frac{\partial\theta}{\partial t}\right)\frac{\partial s}{\partial\phi}$$

$$+\left(\frac{\sin\phi\cos\phi\sin^2\theta}{r}\frac{\partial r}{\partial t}+\cos^2\phi\sin^2\theta\frac{\partial\phi}{\partial t}+\sin\phi\cos\phi\sin\theta\cos\theta\frac{\partial\theta}{\partial t}\right)\frac{\partial s}{\partial\phi}$$

$$+\left(\frac{-\sin\phi\cos\phi}{r}\frac{\partial r}{\partial t}+\sin^2\phi\frac{\partial\phi}{\partial t}\right)\frac{\partial s}{\partial\phi}$$

$$+\left(\frac{-\sin\theta\cos\theta}{r}\frac{\partial r}{\partial t}+\frac{-\cos\phi\sin\theta\cos\theta}{\sin\phi}\frac{\partial\phi}{\partial t}+\sin^2\theta\frac{\partial\theta}{\partial t}\right)\frac{\partial s}{\partial\theta}$$

$$+\left(\frac{\sin\theta\cos\theta}{r}\frac{\partial r}{\partial t}+\frac{\cos\phi\sin\theta\cos\theta}{\sin\phi}\frac{\partial\phi}{\partial t}+\cos^2\theta\frac{\partial\theta}{\partial t}\right)\frac{\partial s}{\partial\theta}$$

$$=\left(\sin^2\phi\cos^2\theta\frac{\partial r}{\partial t}+\sin^2\phi\sin^2\theta\frac{\partial r}{\partial t}+\cos^2\phi\frac{\partial r}{\partial t}\right)\frac{\partial s}{\partial r}$$

$$+\left(r\sin\phi\cos\phi\cos^2\theta\frac{\partial\phi}{\partial t}+r\sin\phi\cos\phi\sin^2\theta\frac{\partial\phi}{\partial t}-r\sin\phi\cos\phi\frac{\partial\phi}{\partial t}\right)\frac{\partial s}{\partial r}$$

$$+\left(r\sin^2\phi\sin\theta\cos\theta\frac{\partial\theta}{\partial t}-r\sin^2\phi\sin\theta\cos\theta\frac{\partial\theta}{\partial t}\right)\frac{\partial s}{\partial r}$$

$$+\left(\frac{\sin\phi\cos\phi\cos^2\theta}{r}\frac{\partial r}{\partial t}+\frac{\sin\phi\cos\phi\sin^2\theta}{r}\frac{\partial r}{\partial t}+\frac{-\sin\phi\cos\phi}{r}\frac{\partial r}{\partial t}\right)\frac{\partial s}{\partial\phi}$$

$$+\left(+\cos^2\phi\cos^2\theta\frac{\partial\phi}{\partial t}+\cos^2\phi\sin^2\theta\frac{\partial\phi}{\partial t}+\sin^2\phi\frac{\partial\phi}{\partial t}\right)\frac{\partial s}{\partial\phi}$$

$$+\left(-\sin\phi\cos\phi\sin\theta\cos\theta\frac{\partial\theta}{\partial t}++\sin\phi\cos\phi\sin\theta\cos\theta\frac{\partial\theta}{\partial t}\right)\frac{\partial s}{\partial\phi}$$

$$+\left(\frac{-\sin\theta\cos\theta}{r}\frac{\partial r}{\partial t}+\frac{\sin\theta\cos\theta}{r}\frac{\partial r}{\partial t}\right)\frac{\partial s}{\partial\theta}$$

$$+\left(+\frac{-\cos\phi\sin\theta\cos\theta}{\sin\phi}\frac{\partial\phi}{\partial t}+\frac{\cos\phi\sin\theta\cos\theta}{\sin\phi}\frac{\partial\phi}{\partial t}\right)\frac{\partial s}{\partial\theta}$$

$$+\left(\sin^2\theta\frac{\partial\theta}{\partial t}+\cos^2\theta\frac{\partial\theta}{\partial t}\right)\frac{\partial s}{\partial\theta}$$

$$=\frac{\partial r}{\partial t}\frac{\partial s}{\partial r}+\frac{\partial\phi}{\partial t}\frac{\partial s}{\partial\phi}+\frac{\partial\theta}{\partial t}\frac{\partial s}{\partial\theta}$$

球座標における速度成分の定義

$$v_r = \frac{\partial r}{\partial t}$$

$$v_\phi = r\frac{\partial \phi}{\partial t}$$

$$v_\theta = r\sin\phi\frac{\partial \theta}{\partial t}$$

移流項の速度次元成分を用いた整理

$$u\frac{\partial s}{\partial x}+v\frac{\partial s}{\partial y}+w\frac{\partial s}{\partial z}$$

$$=\frac{\partial r}{\partial t}\frac{\partial s}{\partial r}+\frac{\partial \phi}{\partial t}\frac{\partial s}{\partial \phi}+\frac{\partial \theta}{\partial t}\frac{\partial s}{\partial \theta}$$

$$=v_r\frac{\partial s}{\partial r}+\frac{v_\phi}{r}\frac{\partial s}{\partial \phi}+\frac{v_\theta}{r\sin\phi}\frac{\partial s}{\partial \theta}$$

ラプラシアンの変換

$$\frac{\partial^2 s}{\partial x^2}+\frac{\partial^2 s}{\partial y^2}+\frac{\partial^2 s}{\partial z^2}$$

$$=\sin^2\phi\cos^2\theta\frac{\partial^2 s}{\partial r^2}+\frac{\cos^2\phi\cos^2\theta}{r^2}\frac{\partial^2 s}{\partial \phi^2}+\frac{\sin^2\theta}{r^2\sin^2\phi}\frac{\partial^2 s}{\partial \theta^2}$$

$$+\frac{2\sin\phi\cos\phi\cos^2\theta}{r}\frac{\partial^2 s}{\partial r\partial \phi}+\frac{-2\sin\theta\cos\theta}{r}\frac{\partial^2 s}{\partial r\partial \theta}+\frac{-2\cos\phi\sin\theta\cos\theta}{r^2\sin\phi}\frac{\partial^2 s}{\partial \phi\partial \theta}$$

$$+\frac{\cos^2\theta\cos^2\phi+\sin^2\theta}{r}\frac{\partial s}{\partial r}+\frac{-2\cos^2\theta\sin^2\phi\cos\phi+\sin^2\theta\cos\phi}{r^2\sin\phi}\frac{\partial s}{\partial \phi}+\frac{2\sin\theta\cos\theta}{r^2\sin^2\phi}\frac{\partial s}{\partial \theta}$$

$$+\sin^2\phi\sin^2\theta\frac{\partial^2 s}{\partial r^2}+\frac{\cos^2\phi\sin^2\theta}{r^2}\frac{\partial^2 s}{\partial \phi^2}+\frac{\cos^2\theta}{r^2\sin^2\phi}\frac{\partial^2 s}{\partial \theta^2}$$

$$+\frac{2\sin\phi\cos\phi\sin^2\theta}{r}\frac{\partial^2 s}{\partial r\partial \phi}+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 s}{\partial r\partial \theta}+\frac{2\cos\phi\sin\theta\cos\theta}{r^2\sin\phi}\frac{\partial^2 s}{\partial \phi\partial \theta}$$

$$+\frac{\cos^2\phi\sin^2\theta+\cos^2\theta}{r}\frac{\partial s}{\partial r}+\frac{-2\sin^2\phi\cos\phi\sin^2\theta+\cos\phi\cos^2\theta}{r^2\sin\phi}\frac{\partial s}{\partial \phi}+\frac{-2\sin\theta\cos\theta}{r^2\sin^2\phi}\frac{\partial s}{\partial \theta}$$

$$+\cos^2\phi\frac{\partial^2 s}{\partial r^2}+\frac{\sin^2\phi}{r^2}\frac{\partial^2 s}{\partial \phi^2}+\frac{-2\sin\phi\cos\phi}{r}\frac{\partial^2 s}{\partial r\partial \phi}+\frac{\sin^2\phi}{r}\frac{\partial s}{\partial r}+\frac{2\sin\phi\cos\phi}{r^2}\frac{\partial s}{\partial \phi}$$

$$=\sin^2\phi\cos^2\theta\frac{\partial^2 s}{\partial r^2}+\sin^2\phi\sin^2\theta\frac{\partial^2 s}{\partial r^2}+\cos^2\phi\frac{\partial^2 s}{\partial r^2}$$

$$+\frac{\cos^2\phi\cos^2\theta}{r^2}\frac{\partial^2 s}{\partial\phi^2}+\frac{\cos^2\phi\sin^2\theta}{r^2}\frac{\partial^2 s}{\partial\phi^2}+\frac{\sin^2\phi}{r^2}\frac{\partial^2 s}{\partial\phi^2}$$

$$+\frac{\sin^2\theta}{r^2\sin^2\phi}\frac{\partial^2 s}{\partial\theta^2}+\frac{\cos^2\theta}{r^2\sin^2\phi}\frac{\partial^2 s}{\partial\theta^2}$$

$$+\frac{2\sin\phi\cos\phi\cos^2\theta}{r}\frac{\partial^2 s}{\partial r\partial\phi}+\frac{2\sin\phi\cos\phi\sin^2\theta}{r}\frac{\partial^2 s}{\partial r\partial\phi}+\frac{-2\sin\phi\cos\phi}{r}\frac{\partial^2 s}{\partial r\partial\phi}$$

$$+\frac{-2\sin\theta\cos\theta}{r}\frac{\partial^2 s}{\partial r\partial\theta}+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 s}{\partial r\partial\theta}$$

$$+\frac{-2\cos\phi\sin\theta\cos\theta}{r^2\sin\phi}\frac{\partial^2 s}{\partial\phi\partial\theta}+\frac{2\cos\phi\sin\theta\cos\theta}{r^2\sin\phi}\frac{\partial^2 s}{\partial\phi\partial\theta}$$

$$+\frac{\cos^2\theta\cos^2\phi+\sin^2\theta}{r}\frac{\partial s}{\partial r}+\frac{\cos^2\phi\sin^2\theta+\cos^2\theta}{r}\frac{\partial s}{\partial r}+\frac{\sin^2\phi}{r}\frac{\partial s}{\partial r}$$

$$+\frac{-2\cos^2\theta\sin^2\phi\cos\phi+\sin^2\theta\cos\phi}{r^2\sin\phi}\frac{\partial s}{\partial\phi}+\frac{-2\sin^2\phi\cos\phi\sin^2\theta+\cos\phi\cos^2\theta}{r^2\sin\phi}\frac{\partial s}{\partial\phi}+\frac{2\sin\phi\cos\phi}{r^2}\frac{\partial s}{\partial\phi}$$

$$+\frac{2\sin\theta\cos\theta}{r^2\sin^2\phi}\frac{\partial s}{\partial\theta}+\frac{-2\sin\theta\cos\theta}{r^2\sin^2\phi}\frac{\partial s}{\partial\theta}$$

$$=\sin^2\phi\cos^2\theta\frac{\partial^2 s}{\partial r^2}+\sin^2\phi\sin^2\theta\frac{\partial^2 s}{\partial r^2}+\cos^2\phi\frac{\partial^2 s}{\partial r^2}$$

$$+\frac{\cos^2\phi\cos^2\theta}{r^2}\frac{\partial^2 s}{\partial\phi^2}+\frac{\cos^2\phi\sin^2\theta}{r^2}\frac{\partial^2 s}{\partial\phi^2}+\frac{\sin^2\phi}{r^2}\frac{\partial^2 s}{\partial\phi^2}$$

$$+\frac{\sin^2\theta}{r^2\sin^2\phi}\frac{\partial^2 s}{\partial\theta^2}+\frac{\cos^2\theta}{r^2\sin^2\phi}\frac{\partial^2 s}{\partial\theta^2}$$

$$+\frac{2\sin\phi\cos\phi\cos^2\theta}{r}\frac{\partial^2 s}{\partial r\partial\phi}+\frac{2\sin\phi\cos\phi\sin^2\theta}{r}\frac{\partial^2 s}{\partial r\partial\phi}+\frac{-2\sin\phi\cos\phi}{r}\frac{\partial^2 s}{\partial r\partial\phi}$$

$$+\frac{-2\sin\theta\cos\theta}{r}\frac{\partial^2 s}{\partial r\partial\theta}+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 s}{\partial r\partial\theta}$$

$$+\frac{-2\cos\phi\sin\theta\cos\theta}{r^2\sin\phi}\frac{\partial^2 s}{\partial\phi\partial\theta}+\frac{2\cos\phi\sin\theta\cos\theta}{r^2\sin\phi}\frac{\partial^2 s}{\partial\phi\partial\theta}$$

$$+\frac{\cos^2\theta\cos^2\phi+\sin^2\theta}{r}\frac{\partial s}{\partial r}+\frac{\cos^2\phi\sin^2\theta+\cos^2\theta}{r}\frac{\partial s}{\partial r}+\frac{\sin^2\phi}{r}\frac{\partial s}{\partial r}$$

$$+\frac{-2\cos^2\theta\sin^2\phi\cos\phi+\sin^2\theta\cos\phi}{r^2\sin\phi}\frac{\partial s}{\partial\phi}+\frac{-2\sin^2\phi\cos\phi\sin^2\theta+\cos\phi\cos^2\theta}{r^2\sin\phi}\frac{\partial s}{\partial\phi}+\frac{2\sin\phi\cos\phi}{r^2}\frac{\partial s}{\partial\phi}$$

$$+\frac{2\sin\theta\cos\theta}{r^2\sin^2\phi}\frac{\partial s}{\partial\theta}+\frac{-2\sin\theta\cos\theta}{r^2\sin^2\phi}\frac{\partial s}{\partial\theta}$$

$$=\left(\sin^2\phi\cos^2\theta+\sin^2\phi\sin^2\theta+\cos^2\phi\right)\frac{\partial^2 s}{\partial r^2}$$

$$+\left(\frac{\cos^2\phi\cos^2\theta}{r^2}+\frac{\cos^2\phi\sin^2\theta}{r^2}+\frac{\sin^2\phi}{r^2}\right)\frac{\partial^2 s}{\partial\phi^2}$$

$$+\left(\frac{\sin^2\theta}{r^2\sin^2\phi}+\frac{\cos^2\theta}{r^2\sin^2\phi}\right)\frac{\partial^2 s}{\partial\theta^2}$$

$$+\left(\frac{2\sin\phi\cos\phi\cos^2\theta}{r}+\frac{2\sin\phi\cos\phi\sin^2\theta}{r}+\frac{-2\sin\phi\cos\phi}{r}\right)\frac{\partial^2 s}{\partial r\partial\phi}$$

$$+\left(\frac{-2\sin\theta\cos\theta}{r}+\frac{2\sin\theta\cos\theta}{r}\right)\frac{\partial^2 s}{\partial r\partial\theta}$$

$$+\left(\frac{-2\cos\phi\sin\theta\cos\theta}{r^2\sin\phi}+\frac{2\cos\phi\sin\theta\cos\theta}{r^2\sin\phi}\right)\frac{\partial^2 s}{\partial\phi\partial\theta}$$

$$+\left(\frac{\cos^2\theta\cos^2\phi+\sin^2\theta}{r}+\frac{\cos^2\phi\sin^2\theta+\cos^2\theta}{r}+\frac{\sin^2\phi}{r}\right)\frac{\partial s}{\partial r}$$

$$+\left(\frac{-2\cos^2\theta\sin^2\phi\cos\phi+\sin^2\theta\cos\phi}{r^2\sin\phi}+\frac{-2\sin^2\phi\cos\phi\sin^2\theta+\cos\phi\cos^2\theta}{r^2\sin\phi}+\frac{2\sin\phi\cos\phi}{r^2}\right)\frac{\partial s}{\partial\phi}$$

$$+\left(\frac{2\sin\theta\cos\theta}{r^2\sin^2\phi}+\frac{-2\sin\theta\cos\theta}{r^2\sin^2\phi}\right)\frac{\partial s}{\partial\theta}$$

$$=(1)\frac{\partial^2 s}{\partial r^2}+\left(\frac{1}{r^2}\right)\frac{\partial^2 s}{\partial\phi^2}+\left(\frac{1}{r^2\sin^2\phi}\right)\frac{\partial^2 s}{\partial\theta^2}$$

$$+(0)\frac{\partial^2 s}{\partial r\partial\phi}+(0)\frac{\partial^2 s}{\partial r\partial\theta}+(0)\frac{\partial^2 s}{\partial\phi\partial\theta}$$

$$+\left(\frac{2}{r}\right)\frac{\partial s}{\partial r}+\left(\frac{\cos\phi}{r^2\sin\phi}\right)\frac{\partial s}{\partial\phi}+(0)\frac{\partial s}{\partial\theta}$$

$$=\frac{\partial^2 s}{\partial r^2}+\left(\frac{2}{r}\right)\frac{\partial s}{\partial r}+\left(\frac{1}{r^2}\right)\frac{\partial^2 s}{\partial\phi^2}+\left(\frac{\cos\phi}{r^2\sin\phi}\right)\frac{\partial s}{\partial\phi}+\left(\frac{1}{r^2\sin^2\phi}\right)\frac{\partial^2 s}{\partial\theta^2}$$

$$=\frac{1}{r^2}\left\{r^2\frac{\partial^2 s}{\partial r^2}+2r\frac{\partial s}{\partial r}\right\}+\frac{1}{r^2\sin\phi}\left\{\sin\phi\frac{\partial^2 s}{\partial\phi^2}+\cos\phi\frac{\partial s}{\partial\phi}\right\}+\frac{1}{r^2\sin^2\phi}\frac{\partial^2 s}{\partial\theta^2}$$

$$=\frac{1}{r^2}\left(r^2\frac{\partial s}{\partial r}\right)+\frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial s}{\partial\phi}\right)+\frac{1}{r^2\sin^2\phi}\frac{\partial^2 s}{\partial\theta^2}$$

３.１３　内径 50 mm の円管を密度 950 kg/m^3 の油が毎分 0.072 m^3 の割合で一様に流れているものとすれば、このときのレイノルズ数はいくらか。ただし、この油の粘性係数は μ=0.038 Pa s とする。

解）管内流れのレイノルズ数の定義は式(3.43)で与えられる。

最初に関連する値をＳＩ単位の基本単位とその組み合わせによる単位であらわしてお

く。

直径　$d = 50\ \ \mathrm{mm} = 0.05\ \ \mathrm{m}$

体積流量　$V = 0.72\ \ \mathrm{m^3/min} = 1.2 \times 10^{-2}\,\mathrm{m^3}/s$

密度　$\rho = 950\ \ \mathrm{kg\,m^{-3}}$

粘度　$\mu = 0.038\ \ \mathrm{Pa\,s}$

管の断面積Sは

$$S = \frac{\pi d^2}{4} = \frac{(3.1416)(0.05)^2}{4} = 1.96 \times 10^{-3}\,\mathrm{m}^2$$

管内平均流速は$u_m = \dfrac{V}{S} = \dfrac{1.2 \times 10^{-2}}{1.96 \times 10^{-3}} = 6.11 \times 10^{-1}\,\mathrm{m/s}$

動粘性係数は

$$\nu = \frac{\mu}{\rho} = \frac{0.038}{950} = 4.00 \times 10^{-5}\,\mathrm{m}^2/s$$

レイノルズ数Re_dは

$$Re_d = \frac{u_m d}{\nu} = \frac{(6.11 \times 10^{-1})(0.05)}{4.00 \times 10^{-5}} = 764$$

３.１４　直径 5 cm の円管内に 100 cm³/min で水を流したとき、流れが層流か乱流か判断せよ。

解）式(3.43)を用いて判断する。最初に関連する値をＳＩ単位の基本単位とその組み合わせによる単位であらわしておく。

直径$d = 5\ \ \mathrm{cm} = 0.05\ \ \mathrm{m}$

体積流量$V = 100\ \ \mathrm{cm^3/min} = 1.67 \times 10^{-6}\,\mathrm{m^3}/s$

管の断面積Sは

$$S = \frac{\pi d^2}{4} = \frac{(3.1416)(0.05)^2}{4} = 1.96 \times 10^{-3}\,\mathrm{m}^2$$

管内平均流速は $u_m = \dfrac{V}{S} = \dfrac{1.67\times10^{-6}}{1.96\times10^{-3}} = 8.49\times10^{-4}\,\mathrm{m/s}$

温度が指定されていないので動粘性係数は 20℃の値を用いて

$\nu = 1.00\times10^{-6}\,\mathrm{m}^2/s$

レイノルズ数 Re_d は

$$Re_d = \frac{u_m d}{\nu} = \frac{(8.49\times10^{-4})(0.05)}{1.00\times10^{-6}} = 42.3 < 2{,}300$$

よって層流。

３.１５　内径 0.1m の管内に 70℃の水を 500kg/h で流す。この場合の発達した流れが層流であるか乱流であるかを判定せよ。

解）式(3.43)を用いて判断する。最初に関連する値をＳＩ単位の基本単位とその組み合わせによる単位であらわしておく。

直径 $d = 0.1\ \mathrm{m}$

質量流量 $w = 500\ \mathrm{kg/h} = 0.1389\ \mathrm{kg}/s$

P198 の付表 3 より 70℃の水の密度は 977.8 kg/m³

P198 の付表 3 より 70℃の水の動粘性係数は 0.4131 mm²/s=4.131×10⁻⁷ m³/s

よって、

体積流量 $V = \dfrac{0.1389}{977.8} = 1.420\times10^{-4}\,\mathrm{m}^3/s$

管の断面積 S は

$$S = \frac{\pi d^2}{4} = \frac{(3.1416)(0.1)^2}{4} = 7.854\times10^{-3}\,\mathrm{m}^2$$

管内平均流速は $u_m = \dfrac{V}{S} = \dfrac{1.420\times10^{-4}}{7.854\times10^{-3}} = 1.809\times10^{-2}\,\mathrm{m/s}$

レイノルズ数 Re_d は

$$Re_d = \frac{u_m d}{\nu} = \frac{(1.809\times10^{-2})(0.1)}{4.131\times10^{-7}} = 4380 > 2{,}300$$

よって乱流。